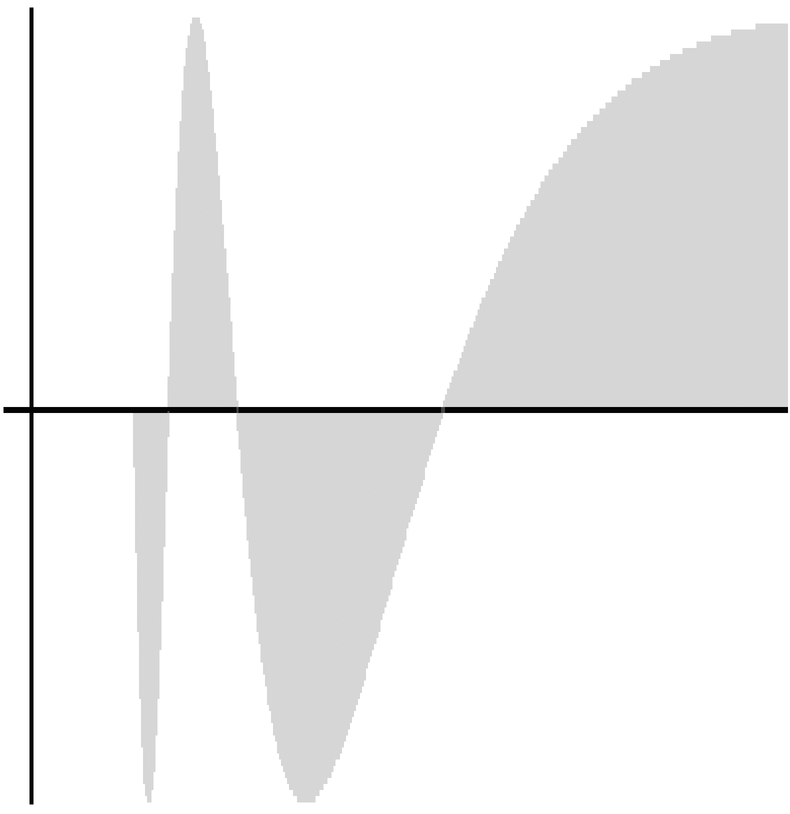
Infinity or not?

An Arithmetical Satire



by Valery Chalidze

Infinity or not? An Arithmetical Satire

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SUMMARY:

Infinity Or Not? An Arithmetical Satire
by Valery Chalidze

This work is a satire on the traditional perception of infinity in contemporary mainstream mathe-matics. Although the author sympathizes with logical constructivism and intuitionism, he stands alone in showing a para-logical tolerance of those who produced the basis for today's infinite set theory and some branches of meta-mathematics.

The causticity of the text does not interfere with the clear logical presentation of such crucial points as proof of the countability of real numbers, and demonstration that the well-known Continuum Hypothesis does not make sense because it is impossible to cover the interval of a line continuously by any quantity of points. The author ridicules the naive and never-proven notion that a line is made of points, explaining that a line is a continuous entity of its own kind. Niether God nor Euclid made a line from points; it is Zeno's curse that some people perceive a line as an infinity of points. With acerbity toward the unrestricted and often fuzzy use of the infinity concept, and care toward the logic of arithmetic, the author leaves no doubt that tradition misrepresents one of the basic entities in the theory of numbers: the periodic fraction. Despite sarcasm on almost every page, the author is still appreciative of those who produced, or tolerate, the cracks in the foundation of mathematics because, at the same time, they were good mathematicians.

In the spirit of reconciliation the author states: "That is not to say that I have anything against para-logicians socially. Some of my best friends are para-logical, and I would even defend their right to practice logic without a license."

By the same author

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MASS AND ELECTRIC CHARGE IN THE VORTEX THEORY OF MATTER

The book presents foundations of the vortex theory of matter, and demonstrates that the basic properties of photons and particles can be described in a philosophical framework of classical physics with certain suppositions about the nature of physical space.

These days applying vortex models to elementary particles looks more reasonable. Twisted vortex rings with left and right rotation in this theory are models of particles and anti-particles.

Maxwell's equations are accepted as the actual basis for the description of aether's kinematics. As to the dynamics of aether, it is shown that Newtonian mass of a particle depends not on the quantity of any substance but on the intensity of rotation of its twisted vortex ring. The non-linear inertial property of aether is presumed; the linear dependence of a photon's energy of its angular velocity follows.

The hypothesis of the topological identity of an electron's and proton's rings lead to value of mass for μ particle which is close to experimental. A model of electric field as field of vorticular filaments is presented. It is shown that the squer of an elementary charge is proportional to the Plank constant and speed of light.

ENTROPY DEMYSTIFIED: POTENTIAL ORDER, LIFE AND MONEY

Whenever we are dealing with matter and energy be it heat machines, biology, economy or the use of natural resources, we must take into account the second law of thermodynamics, which states that the level of disorder (entropy) in an enclosed system can not decrease and that one has to spend energy to decrease disorder in any part of the system. The processes of life and social life are characterized by increasing local order, but are still subject to limitation as dictated by the second law. This brought scientists to the development of the physics of open systems thanks to the ideas of Schrodinger, Prigogine and others. Now we understand that the world is a place where destructive tendencies coexist with creative forces.

What are the inevitable consequences of the fact that we are built from matter, and how much our willing - together with instinctive - behavior is defined and limited by the laws of physics? Limitations imposed on life, social life, economics and the use of environment by the second law of thermodynamics are particularly interesting.

Contents

Warning:

Readers are presumed to be not irreversibly brain-washed by the contemporary theology of the infinite. The word "number" here means a positive number. Three dots in positional presentations of fractions means "so on to infinity", or it designates the infinite part of a set.

The Poor Man's Ivory Tower

Too many interesting things already belong only to narrow groups of professionals: the man on the street -- or from the woods -- wouldn't dare even to approach the temples of intellectual achievement in most areas of knowledge. Arithmetic, however, belongs to each and every one of us, and it is our duty to protect it from subversive mystical concepts. I feel it is my right as a citizen to master and defend arithmetic -- this bastion of impassionate reality in an ever more fantastic intellectual world.

Being somewhat familiar with the clever and elaborate works on the foundations of mathematics, I can see where it is headed: common sense is gradually being replaced by logic for the privileged few, logic of a barely-digestible complexity in which even simple and tested arithmetic has already almost been declared unreliable. The complexity is such that – as in medieval trials by water or fire -- the final logical judgment of guilt or innocence, of right or wrong, is deliberately left to machines, real or imaginary. *Diabolus ex machina* now presides over this logical sabbath.

It is time for common sense to make some declarations. Humans are well known throughout centuries for creating elaborate theories, and then watching those theories crumble or be abandoned or changed due to a real mess in the initial assumptions. Building itself is not too difficult, but it is hard to find firm ground on which to build -- take it from an experienced builder. Common sense often tries to play the role of watchdog, not that theoreticians pay attention to its barking. Common sense also might err, but it was with us before any theories appeared. Our survival itself is evidence in its support. One should trust to common sense sometimes, even if one calls it by the fancy word "intuition".

Introduction To Conspiracy Theory

Why can’t the subversive elements among the self-proclaimed *sapiens* species be satisfied with scientific mysticism in areas more remote from our everyday experience? Let them talk about gravitational black holes somewhere, perhaps eating theoreticians on some planets as we speak. Let them talk about the origin of the Universe (as if they know what the Universe is). Let them fantasize about skeletons in their subconscious and suppressed embryonic memories. Let them develop numerous religious concepts which, traditionally, did not interfere in manipulation with numbers (putting aside the fuzzy unity of the trinity). Even theories that the world might turn upside-down if nobody is looking, does not bother me too much, because I hope that someone is always looking.

I am not the first, of course. Other thinkers, more serious than I am, tried to defend arithmetic from those who picture the world in which we live as even more fantastic than it is.

Our ancestors invented the concept of numbers to count things – a simple and valuable task, especially if you have many children from many wives. But later so many more numbers were invented that counting the numbers became a problem! If one thinks it is sacrilegious of me to joke about mathematics, one should hear the best joke yet: Cantor's declaration that numbers are uncountable. That means non-denumerable, out of this world, imperceptible, lost in infinity. That also means "not really for human consumption," as we humans cannot embrace the uncountable. The century of bewilderment in logical dealing with uncountability speaks for itself. Many people still do not know if they should laugh at that joke or remodel their brain in trying to understand it.

Cantor did not damage arithmetic directly, but without the myth of the uncountability of numbers and the paradoxes of set theory, arithmetic would not be subject to the worst insult in its history: the suspicion that it is incomplete and, God forbid, even inconsistent. (People joke more often about lawyers than about mathematicians, but look: so many lawyers are around and not one of them ever used foggy rumors and seductive misconceptions about Godel's theorems of incompleteness as a defense against embezzlement charges. Lawyers must be more sober thinking than logicians.)

Homage To Kronecker

The name of Kronecker comes to mind as a defender of the arithmetical faith. Maybe he went a little too far; that is what ideologists do. He tried to preach a down-to-earth approach in mathematics, but the time was wrong -- in fact, very wrong. In the late nineteenth and early twentieth centuries there was a wave of scientifically-formulated mysticism in many areas of knowledge, from psychology to physics and mathematics, and then even to logic. This was a healthy but misguided desire to jump in over their own heads, so to speak. Perhaps various social changes and the deterioration of old religious beliefs, led intellectuals to try and revise everything that had been considered the basis of knowledge before. I would deviate from arithmetic too far if I start to talk about the weirdness in the social philosophy of the same period. I myself survived political experimentation brought about by some of those weird theories, and I am among a very few who managed to come out of it with a clear head. I understand that such a statement is not only immodest but, as a self-reference, might lead to a paradox.

And then there was Hilbert who, as a formalist, would be expected to support the forces of anti-mysticism. Philosophically, however, he was a rather funny fellow: he wanted the best of both worlds. Embracing whatever infinity he could find, he also enthusiastically wanted to build a logical basis for mathematics by none other than finitistic methods. Well, it hasn't worked yet, although he himself was instrumental in providing a logical basis to some parts of mathematics.

Knights Are On the Field But I Am Having Fun

The ideological struggle, as far as the foundations of mathematics goes, is still with us since the time of Kronecker. Intuitionism and/or constructivism are the best-known currents that actually try to rebuild mathematics according to their prescriptions on how to avoid mysticism or even some innocent paradoxes. Unlike them, I am just passing through, trying to think and to have fun, not standing on the platform of any ...ism (well, maybe a touch of "be-carefulism" as far as infinity goes). Don't tell me this position is wrong; I believe I invented this genre of arithmetical satire, so I do as I please.

As to having fun, why not? All these centuries of philosophical struggle over "infinity or not" didn't touch the foundation of our life. Mathematics is blossoming and supplying intellectual tools for science, technology and even space exploration (as long as they remember not to mix inches and meters). So, with infinity not affecting the core of our lives yet, I feel free to take it not too seriously (but I will show my teeth in defense of arithmetic.)

In the same fun-seeking way I can imagine having a friendly conversation about *απειρον* and the non-existence of zero with those playful Greeks in the shade of columns around the Agora. Those guys were talking freely and not really under pressure of peer reviews. I know, they wouldn't let me smoke a good cigar in a public place like the Agora, but maybe I would be sipping some diluted wine with them from an amphora hidden in brown papyrus bag. Those guys invented sparkling idle ways to play with words, and came up with plenty of interesting things in the process, including the rules of logic.

I also can imagine annoying Archimedes with quotations from the Bible about counting dust, and with my silly questions about his own fascination with counting the grains of sand in the Universe (well, the universe actually; it has gotten much larger in our time -- expansion, you know, and all the other factors).

In the middle and not so middle ages, in the same playful mood I could offer some spiritually poisonous comments here or there to those righteous logicians in monasteries, but who would let me in? I hear that in those days they had a habit of frying unruly thinkers.

I can even imagine sharing an interest in comparing infinities with Galileo and telling him about my diagonal method (why can't I imagine this fellow to be young, only old and tired?).

Imagination fails me if I try to fantasize about having fun with Dedekind or Weierstrass and others smarties of that time. I feel they were already too serious to tolerate my company, and that goes especially for my contemporaries. Those who make a profession out of what is supposed to be pleasant thinking about the basis of human knowledge, are often intolerable to the extent of killing all the fun. Some of them assume that you don't have anything better to do than sit in the library and read boring logical theorems they have proved, or read about models they have developed. Above all, most of them are not able to express their views in any plain human language -- and they are so proud of it. Many still hold the naive belief that formalized languages can save us from the vicious circle of knowing things only because we know many other things.

This is a good place to state that despite my disagreement, I don't want to make anyone look silly, except maybe myself a bit. Let’s remember that those with whom I disagree did plenty for mathematics and logic.

Paradox Of Self-Counting

It’s been a long time, there have been many thinkers, and we still don't know whether we should touch infinity. We don't know if we need infinity. No rush, plenty of time. If infinity is around the corner, we will not miss it. Winston Churchill once saw infinity, but it was after dinner, so he "let it pass". Others, after seeing the ghost of infinity, did not let it pass, but made all sorts of theories out of it. Like a ball of snakes, these theories became entangled with each other.

Then, as serpents do, they crawled to all possible corners of mathematical space, to the point where it is now difficult to find a mathematical book free of nightmares about infinity. I should be embarrassed: perhaps the "snakes" example is too graphic for my female readers, if there are any; most females have been terrified by snakes since our foremother socialized with one (although that one had legs).

The one hope for resolution of the infinity puzzle in arithmetic I do have, however, is the self-counting property of numbers.

Infinity of everything else would create almost no problem. Tell me there are an infinite quantity of particles in what you consider to be the Universe, and I'll say "Well, it is a question for verification." I wouldn't lose sleep over it.

But numbers! The amazing thing about natural numbers is that unlike anything else in the world, they are self-counting. You call "31" and he answers: "Here! I am the thirty-first." Each number knows which -st, -nd, or -th it is. So if one day we manage to come to an infinite number, it will proclaim: "Here, I am infiniteth".

Until that happens, we will be puzzled with this paradox. When some talk about the "infinite quantity" of numbers, they are referring to the overall quantity in the natural-numbers sequence. That sequence indeed grows endlessly, without limit or any consideration for our inability to perceive things that are too large. Perhaps that is the explanation: it is easier for us to say that numbers "go to infinity" than actually to labor through imagining very, very large numbers. In a way, we are freeing ourselves from any responsibility. Infinity might give us the pride of being grown up and able to talk about such huge things, but it is certainly not under our control.

A contradiction follows, though, from the fact that natural numbers are self-counting. There are only as many numbers in a natural sequence as is shown by the last number, which is named by us: there are N numbers from 1 to N. If infinity is the overall quantity of natural numbers, then due to the self-counting property there must be an infinite number at the end of that sequence. There is no end, however, so there is no infinite number at the end. One didn't need to insult the inhabitants of Crete as a group to obtain a paradox. We’ve been on the road to this self-counting paradox since the time we first learned to count our digits. To avoid the appearance of paradox, infinity was proclaimed to be a special number. Well, not really a number but in some way still a number to some extent ... up to a point, that is. Sounds like something you wouldn't expect to hear in mathematics, doesn’t it? But I simply don't take seriously this anarchist nonsense. If the "special number" interacts with usual natural numbers, as we will see in the ω-trick soon, then it is designed to represent a quantity. And if it is a quantity, then it should be so kind as simply to be a number.

Is it a true paradox or am I just silly? Infinite means "without end". It is actually a negative qualitative characteristic of our natural sequence. Why do people try to elevate a negative qualitative characteristic into the status of a positive number? Indeed, there is no end; this was proven in antiquity. They are all finite numbers and, my God, they can get so large!

Let me construct one relatively large number for future use. Let's write the digit 9 repeatedly, one after the other in 12 point typeface along the circumference surrounding the visible Universe … well, let's make it 14 point typeface, so we'll see it better. I hear that the radius of that circumference is about 15 billion light years, so there are a lot of digits in our number *u*. Now, let's take the first *u* prime numbers and multiply them all. We'll get a number *U* which is rather large but perfectly finite -- at least as finite as the Universe.

Simple Should Be Kept Simple

I am using common sense, together with a simple philosophical and mathematical approach. I know, these days common sense is too common for intellectual blue bloods, but when one is jumping in over one's head, it is useful to see what is still on the ground.

In fact, if we humans have long mastered dealing with points, lines, natural numbers and fractions, I don't see a reason for professional theoreticians to monopolize these objects and tell us that we cannot handle them with our common sense. This is the philosophy, at least my philosophy, of Nature: simple stuff should be dealt with using simple concepts. One might be forced to build elaborate theories instead, but there should be really good reasons (or a good grant) for that.

What does the "philosophy of Nature" have to do with supposedly abstract objects like numbers? It was Nature who produced discrete objects, and our ability to perceive them. You cannot hide from Nature in your mathematical lecture hall. You can only manipulate the abstract shadows of what was produced by Nature. The arrogance of some people!

As to the growing necessity to deal with basic things only through professionals, history repeats itself. Remember, people couldn't speak with their own God without professionals, so the Reformation came. Well, there are still priests around for those who wish to speak indirectly with the supposedly mighty, but many people took it into own hands. Maybe I should nail my notes to the door – where is that arithmetical temple?

Later, gradually the idea of *Res publica* transformed into *Res* through the intermediation of lawyers. *Publica* can hardly defend its own rights; we need professionals even to decipher them for us.

Now we need professionals to handle the ideas of lines and numbers. Is a Reformation coming? Perhaps an arithmetical revolt of the *populus*? I am serious. Even if infinity were to become part of everyday life thanks to theoreticians, I would call for citizens' right to handle infinity without shamans; there is already too much paternalism around.

Infinitely-Presented Rational Numbers

Now, to the core of my numerical protest. It is customary to equate common fractions with their decimal representations even when the latter are infinite although periodic, for example 1/3= .333... Putting aside a deliberate approximation, this equating should not be based simply on the fact that the limit of $\Sigma\ 3/10^n$ (n= 1 to ∞) equals 1/3.

Due to the special nature of the limit procedure, this is not a satisfactory reason for a conclusion within arithmetic. Indeed, it is known that the number we get as a result of a limit procedure might be outside of the sequence the limit of which we are seeking. In this case, 1/3 is outside the sequence .3, .33, .333 .... Another example: for the sequence $y=1/2^n$ lim y=0 when $n\rightarrow\infty$, despite the fact that zero is not an element of the sequence y.

In the case of a periodic fraction like $\Sigma\ 3/10^n$ (n= 1 to ∞), equating it with 1/3 means that arithmetic with its logic didn't manage to handle the problem. The concept of equality as it is defined in arithmetic should not be used in such a case.

So, further in this discussion the infinite decimal representation R_{10} of the rational number R is called an infinitely presented rational number, to designate the fact that a common fraction R and its representation (but not expression) by an infinite decimal fraction R_{10} not only constitute different numbers, but also belong to different classes of numbers: finitely presented and infinitely presented rationals.

It is a common passage in textbooks that if the infinite decimal fraction "expresses" a rational number, then it must be periodic. That calls for the following note.

Periodic fractions are often perceived as expressions with a short period. One should remember that periods of fractions in decimal (or any positional) representation might be very long; in fact, limitlessly long. If we expect a natural sequence to continue infinitely, then one cannot put a limit on the length of the period of a fraction.

For fun, find the period of a fraction U/(U+1). And try to think: are there periodic fractions with an infinite period?

Historical Note

Why is the equating of R and the infinitely presented R_{10} widely accepted as obvious, despite the fact that it is clearly *bad arithmetic* to equate fractions with mutually prime denominators like 3 and 10^n even if n is infinite?

There is the Weierstrassian definition: two numbers are equal if they remain inside of any arbitrarily-small rational interval (the words "for all practical purposes" should be added). This definition is comfortable for use in classical calculus, but it is not acceptable when one analyzes a numerical "continuum" with attention to the limitations of its continuity dictated by this or that numerical system -- yes, this is what I am doing here. For needs of calculus, Dedekind and others had to show the continuity of a numerical scale. They had to imagine the existence of some idealized numerical system in which there is a one-to-one correspondence between all points on a line, and numbers. As a preview of coming attractions: Dedekind's theory simply and wrongly assumes that any point of a linear interval corresponds to a certain number without asking the question: does this particular point correspond to a number that can be expressed in a chosen numerical system? By the way, where did all those points on the line come from? A line is not made up of points. We can put them there, but we cannot assume they pre-exist -- that will be explained.

The assumption of Dedekind's cut theory would be correct if numbers are defined as the length of intervals from zero to the point of cut, without consideration for the possibilities of a particular numerical system. But if length has to be expressed in certain numerical systems, that imposes limitations.

Indeed, Dedekind's cut method showed the necessity of the existence of irrational numbers precisely because a numerical system that contains only rational numbers expressed by finite fractions was not sufficient to assure continuity of the numerical scale. The following discussion shows that the representation of irrational numbers by an infinite decimal fraction also does not achieve the goal of providing continuity of the numerical scale. It was a rather premature pronunciation of achieved continuity. What I really don't understand in connection with Dedekind's theory is how I swallowed it when I was student. The uncountability of real numbers -- no, that I didn’t go for; but Dedekind ... I missed a perfectly good chance to annoy my teachers even more.

As it happens, the "infinitely close" representation of numbers in a decimal system is quite acceptable for all known applications of a numerical scale except when we actually want to know how many numbers there are or, in other words, what is the power of the set of all decimal or binary numbers.

More On Bad Arithmetic

As to the bad arithmetic mentioned above, one should be especially careful not to accept the occasional para-logical tolerance of our brain, which certainly evolved to deal with finite objects only. When humans are trying to think about the infinitely large, be it numbers or gods, or the infinitely small, be it numbers or angels, they are at risk of ascribing to such objects or concepts more properties than agreed upon in advance, as was shown by the history of theological and mathematical thought. Accepting the equality 1/3=.333...

means that our inability to express 1/3 through a decimal fraction somehow disappears in an infinitely long process of calculation.

The fact itself that in the decimal calculation of 1/3 we can never reach a result, shows that there is no decimal expression of 1/3.

Let n be an integer. With the integers n and k let $f(n/10^k)$ be a function that equals 1 when argument is an integer and 0 otherwise. Apparently for n= 3, 33, 333 and so on this function returns zero. It would take quite creative logic to prove that lim f = 1 when n=333... with the infinite number of the digit 3 and k being equal to the quantity of those digits minus one. That means that no number expressed by any quantity of the digit 3 is divisible by 10 in any power, and infinity will not help.

Often, infinity is treated as if it has any desired property. For all finite n, the inequality

$$1/3 > \Sigma\ 3/10^k \text{ (k=1 to n)}$$

is easy to prove by induction. There is no reason to expect that infinity can be responsible for transforming inequality into equality unless we are using a jumpy limit procedure that is outside the logic of arithmetic.

Irrational Numbers

The decimal representation of irrational numbers is not of the same nature as of those rational numbers that cannot be expressed by a finite decimal fraction. We can calculate an astronomical number of digits (as is done, I hear, by some curious computers for π). In such calculations, the irrational number *a* is inside the interval $[r_-, r_+]$ where r_- and r_+ are low and high rational decimal approximations of *a*. And what are the infinite decimal fractions r_- and r_+? Apparently they are rationals in infinite decimal representation, and they are usually derived from some rational expressions that one uses to approximate the irrational *a*; what's more, r_- and r_+ are not periodic fractions. Wait a minute. Rationals are supposed to be finite fractions, and if fractions r_- and r_+ at some moment will remember that, they will stop going further into the intimate depth of being infinitely close to each other. What will then happen with our endless procedure of crystallizing the irrational between two rationals?

Strictly speaking -- and I am trying to do literally that -- the irrational number is not reachable by a decimal expression at all, be it finite or infinite, and for that reason does not exist in a decimal system. We can define irrational numbers as a limit of sequences r_- and r_+ but the limit should be expressed by a number. What to do if there is none in any numerical system we use? Sequences of the digits of the numbers r_- and r_+ simply continue into a depth of infinite closeness, giving us a more and more precise approximation of one irrational number, as well as an example of two rational numbers that are not finite fractions and not periodic. Intuitively we still accept the existence of irrational numbers because we know that *a* is somewhere inside the interval known to us, but as far as numbers that can be expressed in a decimal system, irrational numbers are phantom numbers inside the holes in the scale of decimal numbers.

The concept of irrational numbers throughout the centuries was a test for people's ability for real abstract thinking, and they are still useful for that purpose. In the belief that by an infinite approximation procedure, we can actually hit an irrational number and turn interval $[r_- r_+]$ into a point, humans -- or at least many mathematicians -- showed that they fail the test of proper abstract thinking. They showed too much impatience with infinite procedure, and, I would arrogantly say, a lack of understanding that infinite procedure never ends and therefore there is no result unless we declare that we reach the limit by definition. It is easier and less abstract to agree that we will actually never see the result, because the procedure is infinite. It takes an additional abstract effort to accept that the result actually does not exist if it requires an infinite procedure for reaching it. We can estimate it; we can jump to it through limit procedure; but we cannot get it. I see a problem for the anthropology of mathematics here.

So, irrational numbers exist but they cannot be expressed precisely by usual numerical systems. So what? If one wants a precise expression of some irrational numbers like $32^{1/2}$, let him choose his numerical system accordingly -- let’s say, based on $2^{1/2}$, then a majority of natural numbers will be not expressable precisely. So what again?

I want to emphasize that my disagreement with Dedekind's cut theory is purely arithmetical. If for differential analysis people need a custom-made theory of numbers with a continuous scale -- that doesn't bother me. (Not that they will get such a continuous scale anyway.)

One Point Theorem

Traditionally, the possibility to narrow the interval $[r_- r_+]$ into a point occupied by an irrational number is supported by the One Point Theorem, which is based on Cantor's axiom:

Let A_1B_1 , ..., A_nB_n be intervals on a line such that each following is inside of the preceding. Let also a be an arbitrarily small interval. Let us assume that we can find n large enough that $A_nB_n<a$. Then there is a point X on that line that is inside of all A_nB_n.

The One Point Theorem usually is presented as an immediate consequence of this axiom.

There is only one point X which belongs to all A_nB_n.

The following proof is suggested:

If there is another point Y which is also inside all A_nB_n, then the interval XY is inside of all A_nB_n and we can find a higher number n such that A_nB_n will be smaller than XY. That is considered proof that there is only one point X inside all these intervals.

However, the only proven fact here is that we cannot find an independant interval of our choice XY inside the endlessly shrinking A_nB_n. And that is fine: one wouldn't expect to find an independant interval inside a potentially infinitely shrinking interval anyway. I say: let Y be in the middle of the interval [A_n X]. Now we have two points inside of all A_nB_n no matter how small the interval A_nB_n will become (however, [A_n X] are not the same as *a* in Cantor's axiom, but that axiom is not violated by my designation of an additional point).

Interestingly enough, the One Point Theorem is not presented as proof that the interval A_nB_n can disappear and become a point, but impies it; Indeed, if the goal of this theorem were to show that all the intervals have a common interval, one would not call it the one point theorem, but the one interval theorem. I have to emphasize that the interval remains an interval, with n going to infinity.

This theorem is perceived as obvious in the context of classical calculus, and it is one more example of a limit procedure. As such, this theorem remains useful to support a procedure of finding the assumed decimal representation of an irrational number; but it should not be used to cover holes in a numerical scale. Simply from the fact that A_nB_n always remains an interval (by definition) and X is a point inside it, it follows that always $A_n < X < B_n$ and never $A_n = X = B_n$ and for that reason X cannot be the only point. I wonder if somebody managed to fool a machine into proving this theorem as is fashionable these days. Machines are probably thinking by now: "humans have alot to answer for."

Wait a minute! Where did that point X come from in the first place? As I will show later, a line is not made out of points; it is a continuous entity of its own kind. So, someone had to put that point there. If not, this theorem reminds me of the supposedly true old story about the Russian bishop who tried to persuade one noble atheist of the existence of God: "Look around you. Look how beautiful this world is. Look at the flowers, listen to those birds. Who else could create such world if not God?" Well, that was a multiple choice question, all right. It is also an example of how dumb people get when they possess the "final truth." In our case, one tries to prove the existence of only one point without proving that there is at least one point.

Reversibility

The belief that we can actually express, and not just represent, an irrational number by narrowing the interval containing that number, has stayed with mathematics for more than a century and by now is well situated in the brains of mathematicians, so I will make one more attempt to illustrate my reasoning. If an infinite number of inclusions turns the interval A_nB_n to a point X or, in the previous notation, if eventually $r_- = a = r_+$ then there must be a reverse possibility of transforming the point *a* back to an interval $[r_-, r_+]$. It is mathematics, after all; there is no law of entropy to limit reversibility. Let's take a point and double it an infinite quantity of times. Actually, I don't know how to double a point, so my suggestion is more in the spirit of sarcasm rather than an attempt to follow the prescriptions of constructivism. Of course I am quite sure that there is no way I can get an interval from a point by this imaginary procedure. (Maybe a topologist can, but how will he get the same interval? Hah!) From the axiomatic point of view, we might decide to introduce an *ad hoc* axiom that would provide for getting something out of nothing. I wouldn't be surprised if somebody will prove that such an axiom is consistent with some theory of very, very infinite sets. (Rigorism for rigorism's sake should be admired specially in cases when unexpected results are proven -- the weirder the better.)

We also cannot get a finite number -- or any number at all -- by multiplying 0 by 2 or by 10 an infinite quantity of times (I suspect even topologists cannot do that, but who knows). That is simply because the definition of zero does not change after applying it an infinite number of times. (I am trying to stay within the Peano axioms here).

Maybe one should accept the reversibility argument as a legitimate method of checking proofs in mathematics. If this test fails, one can look for a reason, and maybe there is a legitimate one. The obvious exception must be for the limit procedure -- that procedure is definitely not reversible when limit A does not belong to the sequence that converges to A. When we mentally jump to A as a limit of sequence we lose information about that sequence; we can't restore the sequence using our knowledge of the limit. Indeed, there are a limitless quantity of sequences with the limit A.

Irreversibility is additional support for the observation that the basis for the traditional belief in the continuity of the numerical scale came from limit procedure, which is not kosher in arithmetic. There is a separate logic for such a procedure; we should be careful not to mix it with the basic logic of arithmetic.

Countability -- Cantor's style proof

The proof of the one point theorem, with infinity taking care of our needs, gave me an idea. There is a purely physiological possibility that the guys manipulating infinity have some special ganglion in their brains that I don't. It would be against my scientific philosophy to try to establish who is normal and who is monster, if those infinity-processing ganglia exist. I am just glad for those who are sufficiently ganglious and can see infinitely further than I can. After all, Mozart obviously had plenty more ganglia for processing sounds then I do and I am not jealous a bit; I simply enjoy the product of his ganglion.

What is more, those who like to dissect dead peoples' brains recently discovered that one physicist had some anatomical peculiarity in his brain that supposedly made him so smart. I hope he agreed in advance to such an intrusion. After all, of all our private parts the brain is the most private.

Even if I am infinitely challenged, I am not ashamed of it. But let me try to mimic the work of those who are perhaps special, if not smarter, and can handle infinite procedures as easily as peeling potatoes. But first, I have a question. How do proofs of theorems through constructing infinite sequences work? From Cantor's proof of the uncountability of the points of a continuum, or from the proof of the one-point theorem, I see that one can build an infinite sequence of intervals with each next one inside of the previous. I assume that infinity will hand us a surprising result: if not an infinite rabbit out of a finite hat, but at least one point instead of an interval. That is mighty! Obviously it is a great achievement compared to the primitive, old-fashioned induction proof which is daring, but not miraculous: it doesn't end with some property or object we did not have before.

Reading books about this stuff, I notice that the author often finishes one proof with an infinite procedure and goes to another. Does that mean that any time, without taking a nap after the long journey, one can go from one infinite procedure to another? If so, why not while proving the same theorem? I am as constructive as the next fellow, but I don't see a reason to forbid that. In the construction business one should use all tools available; otherwise one will never manage to compete with the real constructivists.

Now, what is required for countability? I guess we have to put things into a linear sequence and show that we can start counting. It's easy, actually: just do it in the beginning and assume that it will go to the infinite no-end. We don't keep infinity for nothing. No need for the headache of jumping somewhere: "if it is true for n then it is true for n+1", like in the induction procedure.

Ganglia or not, let me try. After all, it is no achievement to be smart with a perfect brain; one has to try and try again.

To prove: the countability of all points in the interval AB of a line, assuming that the interval is jam-packed with points in such a way that points are covering the interval continuously.

Let’s divide AB by the point A_1 in such a way that A_1 is not A or B and is placed between A and B. Then I divide the interval AA_1 in a similar way by A_2 and so on. Each new point A_k is closer to A then A_{k-1}. As result of this infinite procedure I will have point C_1 which is not A by definition but is just next to A. "Just next" indeed, because it didn't have any choice; it was pushed by my forceful infinite procedure to land next to A, like a crowd in the theater pushes you to take a seat next to someone who is eating smelly popcorn. This was the first step of my building a countable sequence of points which is supposed to cover a continuum [AB]. Not bad, not bad at all. I just started to mimic people with infinite ganglia, and I already arrived at a result unheard of before: I made two points sit together next to each other in a very constructive way.

Now I will divide the interval C_1B with a similar infinite procedure, and will get a point C_2 which is not C_1 but just next to it. And, as they say in smart books authored by almost-constructivists, so on. As a result, I will get a countable sequence AC_1C_2...which after an infinite number of infinite procedures will cover AB with points in such a way that no points will be left uncounted. Am I smart or am I smart? Not only did I learn how to do infinite procedures without the proper ganglia, I also figured out that if one could use it in different theorems one after another, one can use it in the same theorem.

There is still a grammar question, of course. When I talk about infinite procedures, I say "it will do this or that". My German is rusty; did Cantor keep presenting results in the future tense? If so, maybe that future has not yet arrived?

The attentive reader no doubt noticed that the presented proof is complete nonsense, and not even because of the abuse of infinite procedures. From the beginning it is simply impossible to place the point C_1 right next to A. Either the points are separate with an interval between them, or C_1 will be on top of A. That is the kind of creatures they are -- naughty. They don't have length, you see.

Numbers and Holes

As mentioned, I am not declaring some numbers to be non-existent as such, but only inexpressible within a numerical system. Inexpressible numbers can be imagined to be between some expressible numbers, as for example $.999...^{1/2}$ must be between 1 and .999... -- where else?

The opposite position could be declared and justified also: that "numbers are the only entities that could be expressed by some numerical system", but I am a lesser formalist than is needed for that.

A deficit of expressible numbers should not embarrass us. There are functions, and our theoretical constructions, and there are numerical systems. If they do not always agree in the representation of coordinates of some point, it does not mean that the point cannot exist in our minds. We didn't develop the ability of abstract thinking for nothing. The problem with humans is that we construct something abstractly and then we want to touch it, to put it outside the realm of abstraction simply in order to make sure that it is part of reality. But abstract construction is real enough (call me a paleo-platonist if you wish). Of course, sometimes we actually can touch our abstract construction. We can touch the diagonal of a one-inch square despite the irrationality of its length, but we cannot pinpoint it on a numerical scale based on an inch with absolute precision. What I am talking about? When is the last time you used perfectly expressible numbers like the finite

.748291093845762013475674837667528947582187921212092984917283682789393384765249578767883334457544

with absolute precision? We care about the existence and proper place of numbers, not so much about precision; but we can get it to any degree if we want to.

Infinity and Continuity

In addition to the philosophy of numbers, my interest is also directed to understanding the mystery of collective human thinking. Why is it so easy for people to accept that zero does not belong to the sequence $1/2^n$ and at the same time it is commonly accepted that 1/3=.333...?

Logical prescriptions can help only when we have clearly defined the objects and their relations. Infinity and continuity are natural stumbling blocks to logical thinking. As with religion, it should be a matter of freedom of conscience for every theorist to define his own infinity. Relations between infinity and continuity, however, limit freedom of choice.

Indeed, from the point of view of potential infinity, we can imagine an endless decimal calculation of 1/3, the result of which never reaches the actual 1/3. That means that the decimal .333... and 1/3 are different numbers and 1/3 does not exist in a decimal numerical system. That means a breach of continuity; it means that 1/3 is in a hole in the decimal numerical continuum.

On the other hand, if we choose to accept that an infinite process of calculation will actually reach a moment when .333...=1/3, then we'll preserve our belief in continuity, but lose the infinity of the calculation process at the moment when we manage to express 1/3 in a decimal system. Infinite means endless, and infinity is lost if an end in the form of .333...=1/3 is achieved (unless we in a sneaky way assume eternity here as well as infinity. Fine with me.). Mathematical tradition prefers to accept both: the endless .333... which never reaches 1/3, and the equality .333...=1/3 . This duality of beliefs does not shake the illusion of continuity or infinity. Apparently the only logical support for this tradition is the fact that the sequence .3, .33, .333, ...

converges to 1/3, which is correct; but it does not mean that that sequence includes 1/3.

Choosing the Nature of Infinity

The crucial point in choosing the nature of infinity, in my view, is to assure the preservation of the nature of objects and the relations between objects. Zero is the limit of the length of the intervals $[0,1/2^n]$ when $n\rightarrow 8$ but the length of any of those intervals cannot equal zero. That means that careful application of a limit procedure does not violate the nature of objects: the interval remains an interval, the point remains a point, zero remains zero. The limit is not only a point not to cross, but also a point not to step on. When we narrow the interval $[r_-, r_+]$ to infinitely small, we don't change its property of being an interval. It still has two distinct ends, and r_- and r_+ are not equal to each other and not equal to our desired point inside that interval.

Apparently, the preservation of dimensions is involved here. A point has zero dimensions, an interval has a dimension equal to 1; the infinite shortening of an interval cannot change the dimension of an interval simply because it is getting small. Again, from the point of view of potential infinity, changing the dimension if it would occur would be the end of an infinite process of shortening the interval.

One can define, of course, actual infinity as the length of the process (in calculation steps) of shortening an interval until that interval becomes a point. Then r-=a= r+. We'll get a decimal expression, not just a representation, of irrational numbers; that's fine. But that will change the nature of our shrinking object, the interval. It will turn it into a point, it will change the dimension of the object. And that is only one trouble with actual infinity. Do you want to live in a world with objects that have the ability to change dimensions when we wish? Then you should travel: there is no such possibility in this world.

A Trick With Actual Infinity

It is also a common belief that the periodical decimal fraction 0.999...=1. Most textbooks present it as obvious without explanation. The Weierstrassian definition of equality is used by some authors as a reason for declaring it. We also saw that a limit procedure can be used to support statements of this kind.

One might like to compare this .999... number with something. Well, we constructed a large number *U* . If we calculate the fraction U/(U+1), we will get a number quite close to .999... . Not as close as .999... is to 1, but still close.

There is also a direct "proof" that .999...=1, however, and it is connected with the nature of the human perception of infinity. I will show that this perception is contradictory. With x=.999... the direct proof goes as following:

$$10x = 9.999...$$
$$\underline{-x = .999...}$$
$$9x = 9$$

and for that reason x=1.

To me it looks exactly like contradictions are supposed to look, and the advantage of this one is that it is so simple. Indeed, my goal for a detailed invasion into the philosophy of arithmetic originally was to look for the simplest contradictions connected with infinity, as it is precisely infinity that has caused so much trouble for those who have tried to be loyal to logic despite the attacks of mysticism on some intellectual concepts.

Why does the contradiction jump out at us when we are dealing with the simplest arithmetical procedures? Well, there is actual infinity behind those three dots in this case. On the first line the number of digits after the decimal points is actual infinity, which is viewed as a number, and for which the letter ω is often used. The second line is the first line divided by 10, and that makes the number of digits after the decimal point $1+\omega$

The contradiction 1=.999... follows from a special and rather weird rule in dealing with actual infinity. It is called annihilation:

$$1+\omega=\omega.$$

So infinity is not only infinite, it is also not very definite -- to a point of complete freedom of division by a finite number without any chance to change it! And serious people mix it in expressions with usual numbers. Who is writing satire?

I have a feeling that here we have touched on the source of many logical difficulties in the foundations of mathematics: *mixing quantities and qualities*. Infinity is not a number, at least in arithmetic. It is a concept. I might as well write $1+\omega = 2$, meaning that the concept of one plus the concept of infinity gives me two concepts. If one must use infinity as a number in the usual sense, one probably should do it only temporarily and with an apology for weird results on each line. It can be part of what is called in physics a "*gedanken* experiment". Of course in set theory $1+\omega$ has particular meaning that I will not ridicule here. Let set theorists write their own satire.

If we interpret the statement $1+\omega = \omega$ as qualitative, it means that ω does not lose the quality of being infinite if we add 1; that is intuitively acceptable. The same statement declares that 1 loses the quality of being 1, the quality of a separate object, after being added to infinity -- that is understandable but rather bothersome even in a broader philosophical context, especially if one worries about the value and position of an individual element being viewed as a hardly-noticeable part of a much larger entity, be it God or a collective. There are some schools of thought that find dissolving oneself in God, Nature or a perfect society quite desirable. I think that with the annihilation rule, mathematicians set a bad example for other humans.

In any case, $1+\omega = \omega$ is a qualitative statement even if we view it as legitimate. It should not be mixed with statements about quantities for solving arithmetical equations; otherwise we violate the law of the conservation of discrete quantities, which is the base of arithmetic. Declaring ω to be a number in some conditional sense or in a separate theory is still possible, but mixing it with numbers in arithmetic invites contradiction.

Another quite frustrating thing about the ω-trick is also its tolerance to the indefinite nature of infinity. It is ready to swallow anything. Well, maybe this is O.K. for other theories, but not for arithmetic. We want to keep our books in order. So, as we see, for the convenience of the concept of continuity it is declared that the interval 1-.999... is equal to zero.

Silly me, trying to explain something obvious against peoples' better judgment. This case is not logical, it is socio-mathematical. The community of smart people needed continuity of the numerical scale; they proclaimed it and are ready to bully out any simple proofs to the contrary.

The Smallest Decimal Number

If we accept that the interval $1/10^n = 1- 0.999...$ cannot disappear into a point in infinity, and don't simply declare $0.999... = 1$ as tradition demands, then we have an example of the closest numbers, as far as they can be expressed in a decimal system. There is no decimal number, no matter how small, that can be added to .999... without reaching or exceeding 1.

This also means that the number $\varepsilon_{10}=1-.999...$ is the smallest possible number in a decimal numerical system. Are there some real numbers between 1 and .999... about existence of which we can make a theoretical conclusion? Of course, there are an unlimited quantity of them, for example $.999...^{1/2}$ or $(1+.999...)/2$, but they cannot be expressed in a decimal system. And it is natural because $[0, \varepsilon_{10}]$ is *an interval*. We can put an absolutely unlimited quantity of points in any interval and imagine, declare or show that those points correspond to numbers -- but not to numbers expressible in a decimal system.

The decimal system did alot for us. We should give that system a medal, but with ε_{10} we have reached the limit of that system's abilities. Those who declare that ε_{10} equals zero can hope for the limitless ability of the decimal system to express any small numbers, but where will they find those small numbers if they get rid of ε_{10}? (Did anyone check $\Sigma(1/n)^{(1-\varepsilon)}$ for the "all natural n" without assuming that $\varepsilon=0$? Is it converging?)

How small is ε_{10}? Remember our U number. Well 1/U is rather close to ε_{10} by human standards.

Infinitely Many Within "Infinitely Small"

Let's look at the set of points M(x)=max{sin(1/x)} in interval [-π, π]. (See graph of sin(1/x) on the first page, distorted for artistic reasons.) How close are M(x) to each other for small x>0? It is simple and illustrative. Take y=1/x (0<x<π/2) and find all x for which

$$y_{k+1} - y_k = 2\pi$$

(numeration from right to left toward zero). Well, to say "all" is cruel because it is endless, but endless in a nice kind of way because the function is not defined in x=0. Now:

$$1/x_{k+1} - 1/x_k = 2\pi$$

$$(x_k - x_{k+1}) = 2\pi\, x_k\, x_{k+1}$$

So, the distance D between the points of M is proportional to x^2 for a very small x.

Now let's look at my $\varepsilon = 1/10^N$ with N→8. Around x=ε the distance D between the points of M is proportional to $1/\varepsilon^2$.

This is quite a small distance; inside of ε the points of M are much closer than the closest decimal numbers 0 and ε. And they get closer and closer to each other as we move to 0 inside the interval ε.

This is a very interesting case. Sin(1/x) is not defined in x=0 and *sin* is a modest and reliable function. It is not known to produce anything infinite or self-destructive.

Let's figure the quantity of the elements of set M within the first epsilon next to zero. Ha, I got it! There is, so to speak, an infinite quantity of maximums of sin(1/x) close to x=0 inside of ε. What's more, the distances between those points appear to be infinitely small but actually they are always finite, and each of them is well defined and constant! Yes, that is how many intervals we can put on the good old Euclidian continuous line: we can put an infinite quantity of finite intervals and all that is on a tiny interval that many smart people proclaimed to equal zero when they stated that .999...=1.

So we have clashing infinities: an infinitely large quantity of points with finite intervals between them inside of the supposedly infinitely small ε.

The fact that the distances between the points of M(x) are always finite is very important. It simply shows that $1/10^N$ with N→8 is not infinitely small; otherwise it could not contain even one finite interval. Why is this so? Because N is a natural number. It is always finite despite the fact that it grows endlessly. Apparently my ε does not fit into the classical description of an infinitesimal interval, which is supposed to be smaller than any finite interval.

Together with or instead of gifts bearing ganglion, there are probably a few screwed-up neurons in the head of each human who ever took calculus. Those neurons contain deeply-carved nonsense: "0=1/8" and in the same time "infinitely small = 1/8 ". When N grows infinitely, it remains finite. For that reason 1/N and $1/N^{10}$ is always finitely small, never infinitely small.

What is more, where is there a place for the infinitely small if the distances between the points of set M near zero are all finite, well-defined and can become smaller than absolutely anything imaginable?

Of course, all those points of set M are beyond the reach of the decimal system; they are inside the interval ε. Praise the Lord: we are so smart that we can operate with numbers even if there are not enough numbers to express it. There are theoretical ways to present numbers besides numerical systems.

Dealing with π, Archimedes did not see any precise number for it. He knew it was somewhere between his approximations given by the perimeters of an inscribed and a circumscribed polygon. And, after people have fantasized about infinity for centuries, I still say the same about irrationals: that they are between points that can be reached by a numerical system. What was good for Archimedes is fine for me.

On Our Understanding of Numbers

Is it healthy to allow numbers to exist even if they cannot be expressed in some numerical system? The obvious response is: what is a numerical system? Decimal or binary and so on? That is what we use for more or less precise representations of numbers. But what about the theoretical representation of numbers, especially if we know where to put them on the scale of a usual system? If we cannot have irrationals as precisely expressed objects, we at least can make peace with them through formal recognition.

When I started my recent involvement in these problems, which have bothered me since youth, my absent-minded expression produced my wife's question: "What you are in now?"

--Arithmetic.

--I thought it’s done.

--Well, give me the definition of a number, I might do the rest.

And the definition I still don't have. And never will, really, because any definition would be in words and words call for their own definitions. I can say that a number is a symbol with such and such properties. What it would tell me? I can accept that a number is or expresses some set, but that would lead to even more problems. In other words, I can fool myself by giving some working definition but deep down I will know that it is not a real definition.

Anything we define leads to the need for further definitions, and through a chain of attempts sooner or later it will come back to us, even if in a nightmare; and not just come back, but with a demand to be defined again. We are in a vicious circle of words that can be defined only through each other. It’s no wonder that many philosophers, as I assume, prefer to be silent. One even had enough of a sense of humor to raise his voice suggesting silence as a method of communication.

It was said: "In the beginning was the Word..." (John, 1,1) but we don't know which one; we are powerless to give any primary definition.

With numbers, we humans at least have *knowledge through action*: we manipulate numbers in some standard ways and, seeing that manipulation, one can say " Ah, there are numbers!" Indeed, every word we can say about numbers might have other meanings, or together they might be ambiguous; but the ways we deal with numbers, if honestly and without ω-tricks, leave no doubt that we are dealing with numbers.

I remember being very impressed (about a thousand years ago) by a lecture of a Russian geometer, probably Efimov, who demonstrated that separate groups of Euclides' axioms might mean something completely different than what we are accustomed to perceive as points and lines, but together all axioms are applicable exactly to points and lines. So, in a way, Euclides' definitions are not really needed. His axioms leave us no choice but to understand what are points and lines; or, at least, it should be this way. Of course, some systems of axioms might be applicable to more than one model, but one can deal with that and it is even more interesting.

Even if it is not as simple as I expressed it here, still knowledge through unreliable words can be a phantom, yet knowledge through action, through manipulation with objects is reliable enough to open the door to the possibility of further action. It also goes well with thinking through action, which is the way of evolution of life, at least the life around us.

That is why I feel that a clear knowledge of what numbers are is so important, not only practically but philosophically. That is why I try to defend proper ways of manipulating with numbers. Each time we deviate from those proper ways, we actually change the implicit definitions of numbers acquired and reinforced over thousands of years.

Maybe now I am beginning to understand at least one meaning of Kronneker's aphorism, "God created natural numbers, and the rest are made by man." *I am* and *they – numbers -- are,* and this is already a good basis for relations. Natural numbers do not depend on our numerical systems.

So, to answer the previously-posed question, I say, yes, it is healthy to recognize the existence of numbers inexpressible in this or that system even if they are somewhere inside the interval [ε] between the closest numbers that can be expressed in a particular system. As a creation of man they may be not always expressible by a natural system like S_{10} but at least we keep a warm place for them and know where they will jump out if we'll try to find a system more comfortable for them. Change the system from S_{10} to S_1 and 1/3 or 1/2310 will be expressed by a finite fraction. As to the irrationals between infinitely-presented rational numbers in any system S_n, we might for example establish S_π or some other system to accommodate some of them precisely, but it is much more enjoyable to save their theoretical expression and keep them inside the intervals [ε] between rationals, and know their value with arbitrary precision – a clear reminder that those numbers are a creation of man.

The interval $[0, \varepsilon_{10}]$ has the smallest length expressible by a decimal system. From the traditional point of view, this length is infinitely small, of course, but it is clearly described and it is not equal to zero. And we saw already that it is arbitrarily finitely small.

Personally, I feel that the only infinitely small thing is our knowledge of Nature. That is progress since the days of Socrates, who stated that the only thing he knows is that he knows nothing (did he know that he uttered a paradox?)

Different Numerical Systems

If we approximate, then ε_{10}=1-.999... ~ 0.000..001 with the number of zeros after the period depending on the depth of approximation. The 1 at the end will always be there, no matter what depth of approximation we choose.

If we "go to infinity", then ε_{10}=0.000...001 with an infinite number of zeros in the middle. The digit 1 must be there, in infinity. Such an image looks controversial because one might have the impression that I am dealing with a finite decimal fraction if the end of it is known. Later I will show that it is quite natural. Sometimes we can know the end of infinity. In any case, ε_{10} is simply $1/10^n$ with n to infinity.

Now about presenting ε in different numerical systems. The image of ε is the same in any positional system S_p (p>1) be it decimal S_{10} or binary S_2 or a system with any other base number p even irrational:

ε_p= .000 ... 001 or $1/p^n$ where n goes to infinity.

In each system ε_p is the smallest number for that system S_p; but for a different p_1 and p_2, ε_{p1} and ε_{p2} don't represent an interval of the same length (unless we are playing the ω=1+ω game, in which case $\varepsilon_{10}=\varepsilon_{1000000000}$ and so on). To be cautious we should not compare the length of the intervals $[0,\varepsilon_{p1}]$ and $[0,\varepsilon_{p2}]$ in different systems unless we choose to deal with potential infinity and compare $1/p_1^n$ and $1/p_2^n$ for the same n as n going to infinity. In this case ε_p is smaller for larger p.

One should avoid the temptation to choose actual infinity as p with the hope to create the new illusion of making a numerical scale as continuous as a line. But then, who am I to lead you not into temptation?

An Excursion Into the Stone Age

There is a numerical system that is not as comfortable as the positional, but quite natural and useful for presentation in this particular case: S_1 -- a system based on one. In this system there is only one digit -- 1 -- and the number n is presented with a series of n ones as for example 5=11111. One might be reasonably sure that our distant ancestors used this system to establish the main rules of arithmetic at a time when arithmetic was an experimental science. The same system was reincarnated by Peano's axioms where natural numbers are presented as cousins of one so many times removed.

Peano's axioms give the impression that we are building a natural sequence. I wonder if it does not already state the existence of that sequence by assuming that there is a next number and that we can take a step to get to that number. It is that self-counting property of natural sequence again. If we define a number three times removed from 1 we are already counting, we are already using number 3 before we define it. So we really did not define it. What if we would simply state as an axiom "there is a natural sequence" and then infer the properties of numbers, and the rules of manipulation for them? Less economical, but more frank. Of course, one does not have to be a slave of the sequence, and might choose to deal with numbers randomly, postponing putting them in order till better days.

For writing convenience the fraction 1/n I presents as n digits 1 after the period: 1.1=1; 1.11=1/2 and so on. In this system we don't have to imagine a number ε as containing an infinite number of zeros after the period, and having the digit 1 in infinity. It is simply ε_1 =1.111... and imagining this number is not more difficult than imagining the decimal periodic fraction 0.9999... . The good thing about S_1 for positive numbers is that it does not have a zero at all!

Using the S_1 numerical system helps us to imagine the actual infinity E = 111... , if one must, as an infinitely long sequence of digits "1". I probably could use ω instead of E but symbols, like words, are too connected in peoples' mind with what was said before about them and ω already has a questionable reputation. This system makes it clear that if there is an infinite number, then there is only one: none other than E can be expressed in this system. Other systems -- positional -- are derivatives of this "stone age system" and even for that reason one should conclude that in other systems too there is only one actual infinite number, if any, and to be cautious not to accept any infinite sequence of digits as a number. Apparently, ε=1/E = 1.111... is the smallest possible positive number in the system S_1.

System S_1 is, of course, a system of common fractions. It is powerful enough to express any number expressed by a positional system, so if any statement is proven for a positional system, but cannot be proven for S_1 then one must look for the reason. Most likely the reason will be in the misleading properties of that positional system. And if any statement is proven in S_1 then it is so for any system.

Summary of Arguments

I feel I am too talkative, trying to explain many points, but I want it to be understood by the average genius. In a formal presentation it would be enough to falsify a one-point theorem with an explanation that without use of a limit procedure, an interval can never become a point no matter now small it gets.

Yet, as long as I have touched on other problems, I will summarize them now.

The dream of the true formalist is to put a chosen formalized language as well as the rules of inference and axioms into a machine and see it making the theory, so one could be sure that there are no clever human tricks involved. The existence of such a dream is evidence that humans don't trust their own logical ability -- and rightfully so. I am very far from being able to tell a machine how to establish the degree of continuity of a numerical "continuum", so I'll stop now and check to see whether intellectual arrogance or wishful thinking influenced my conclusions. I am human too, you know. At least there is some circumstantial evidence leading to that conclusion.

The crucial point in my presentation of holes in a numerical "continuum" is the statement ε>0 with [ε] being interval serving as a step on the numerical scale. It is well defined within the concept of potential infinity for each numerical system because N or p^N remain finite, even if very large. Number ε should not be equated to zero even if one prefers to use actual infinity. It is not my invention, and it is not a new kind of number. I am simply trying to show that ε-numbers are there if one wants to enforce the rules of arithmetic.

I based my critique of the traditional discarding of ε-numbers on the following reasons:

1. An interval remains an interval. If we split the interval into two parts, those parts are also intervals, no matter how many times this operation is repeated. Infinity cannot change it.

2. Zero remains zero no matter how many times we multiply it by some number, and infinity cannot change it.

3. If a fraction with the denominator q_1 cannot be expressed through a fraction with a denominator q_2^n, the infinity of n cannot change it.

4. The implicit use of a limit procedure in arithmetic often contradicts the logic of arithmetic because the limit procedure may lead to a jump from a sequence to its limit

when the limit does not belong to that sequence. What is more, it makes us jump from an interval to a point, which has different dimensions.

On the Nature of ε-Numbers

All the above reasons should be respected in arithmetic if we want to stay loyal to definitions and rules. This assures me that when dealing with ε-numbers (its own for each S_p) I am not trying to introduce something non-existent in comparison with other numbers, nor to revive the somewhat artificial introduction of infinitesimals of old times.

In order to understand the nature of ε-numbers, let's imagine how we can eliminate them. Declaring them to be zeros is wrong; I believe I showed that. What is another possibility? It is easy to see that ε-numbers exist to the extend of our acceptance of infinity and the possibility of the unlimited use of division by any non-zero number in arithmetic. If we construct a numerical scale without infinity but ending on some large number (equal, let’s say, to the number U presented earlier; and I am sure U will grow together with our presumed knowledge of the Universe) there will be finite ε-numbers which in each positional numerical system equal .000..001 with a large but finite number of zeros. Theoretically with so large a "largest" number U we'll get a test of infinity without the troubles that come with it.

However, if we accept infinity, then "infinitely small" ε-numbers are unavoidable. They are simply on the other end of the concept of infinity: they are only as indefinitely small as infinity is indefinitely large. One should not play with large infinity without taking this into account. Yet I showed earlier that an "infinitely small" ε can host plenty of finite intervals. (It is easy to demonstrate also that there are some finite intervals which contain some infinite intervals as well, but the margins on my screen are too small to show it with illustrations.)

The ancient Greeks would understand me better: they were lucky not to have zero -- this negation of numbers which is declared to be a number. Numbers for Greeks were the length of intervals, as they are supposed to be if one arrives at numbers from the ideas of geometry. The smallest interval naturally complements the largest interval if we observe the symmetry of large and small around point one on a numerical scale. With my contemporaries who are accustomed to zero and know that lim 1/n = 0 when n goes to infinity, it is easy to be fooled into accepting that one divided by infinity is zero. Unlike the Greeks, they don't keep in mind the relations of numbers to intervals.

I agree that in the framework of potential infinity, intervals defined by ε-numbers produce mental parallels to what is known in classical calculus as infinitely small. I also agree that if we accept actual infinity, then ε-numbers might lead to the danger of contradictions; but actual infinity does not need any help in order to lead us to contradictions. Personally, I see the sequence of natural numbers as an endless sequence of finite numbers, and infinity in my understanding is a quality, not a number of any kind. And I definitely disagree that ε-numbers should be declared to be zeros for the sake of preserving the illusion of continuity of the numerical scale. In any other theory depending on needs and agreements, please do this and enjoy. Not in arithmetic where no numbers should be left unaccounted for. Also, there should be no illusions in arithmetic.

Reference to an assumed numerical system of the stone age was quite appropriate in this text, as during that era of human development arithmetic most likely was science, based on the observed conservation law for discrete matter. And to this day, things should not simply appear and disappear in arithmetic; that should be left to more clever mathematical theories, or to the physics of the twentieth century, or to theology.

Filling the [0,1] Interval With Numbers

The irony of my discussion here is that I have to insist on the existence of a traditionally unrecognized class of numbers -- ε-numbers (ε=.000...001 in each S_p) -- in order to show the holes in the numerical scale. What's more, recognizing the existence of holes between numbers makes it possible to show the countability of the numerical "continuum" even if we accept that the infinite number -- actual infinity E -- exists and to some extent can be operated with as a usual number. Indeed, having the minimal number ε next to zero I have the interval [0, ε], with which I can cover the interval [0,1] epsilon by epsilon. While I do this consecutive covering I pass and count every point on that interval, which corresponds to a number on the numerical scale. So, geometrically there is no problem for the countability of the "numerical continuum".

Or I can do it in different manner. If we multiply ε by n=1,2,3... till infinity E we'll get a sequence of real numbers n/E as dense as the numerical system permits. We will reach 1 when n reaches infinity E. That will solve the problem of the countability of numbers of the interval [0,1] using multiplication.

Unlike in the geometrical way of covering the interval [0,1] by [ε], there is an almost crushing fine point in this arithmetical scheme, however. To illustrate it, let's deal with infinity E as if it were an actual number that can be used in arithmetical operations (with apologies!). We got 1 by multiplying ε by E, but how can we get 1/2 if there is no such thing as half of infinity? Well, that will teach us not to put our finite nose where it doesn't belong.

Indeed, if all numbers n *before we reach infinity* are finite, then all n/E in our sequence are "infinitely small." So, when we come to infinity E, it looks like we jump to 1 from the infinitely small numbers in our sequence. Where is 1/2 or 3/13? Note that if, instead of the mysterious E we use any, absolutely any large number, then 3/13 will not play a disappearing act on us! That is why potential infinity is so useful: we know that infinity must be there somewhere, we get a taste of it, but we approach it with ever increasing large numbers and it remains a friendly infinity, one we can perceive with our finite senses, finite logic and, more important, with the axiom of induction. Once we try to jump in over our heads and proclaim that we can master actual infinity, we lose 3/13 and all the other good finite fractions, or have to deal with different infinite numbers (like half of infinity) that are really outside usual healthy arithmetic.

What do humans do with such a warning? Well, some are ready to retreat into the familiar cozy land of tested arithmetic. Others are arrogant enough to create new theories hoping for "paradise" with infinite horizons. I don't blame them for searching, or for arrogance, not at all. I just try to defend what is mine, try to defend comfortable arithmetic.

What actually happened when we lost all finite fractions is the demonstration of the fact that we cannot rich infinity in arithmetic. That is an illustration of the fact that infinity is a quality. We can neutralize the infinitely small with the infinitely large and get a finite number, but we cannot use arithmetic to manipulate qualities instead of numbers.

This illustration that infinity is a quality and not a quantity permits us to avoid weird results like $1+\omega=\omega$. It also answers the well-known trick of a good mathematician but a poor inn-keeper.

Satan provoked David to count the people of Israel and got him into trouble. Maybe now she provoked me to count numbers in the so-called "numerical continuum". It certainly wasn't God, because he was right beside Cantor participating in creating the illusion of the uncountability of numbers.

In any case I denumerated a numerical "continuum" geometrically with no contradiction at all, or with some difficulty if I jump to actual infinity and use it as a number.

The demonstrated countability of the numerical continuum is actually not a big deal. As shown above, that "numerical continuum" is not the same as the one traditionally perceived as a continuum of all points on a line. It is simply a numerical scale provided by any numerical system S_p, and it does not contain irrational numbers because they are not expressible in any S_p if that system is based on natural numbers. Do not rush to jump to the conclusion that if there are no irrationals in my counting, then there was no reason even to start it. Actually, what I counted includes a double quantity of irrationals: remember r_+ and r_-? Well, I counted them. But in Dedekind's traditional "continuum" they are absent. They disappeared into the unity of the trinity and became irrationals: two of them for each irrational!

Important note: holes or not, I showed the countability of the same numerical "continuum" that was declared uncountable. So far I use S_1 but that doesn't matter, because if the decimal system is not in correspondence with S_1 then it is no good in the first place. The difference with the Cantorian picture is that there, the numerical "continuum" was equated by declaration with the continuum of all points on a line. That picture is wrong because there is no one-to-one correspondence between the numbers of any system and the points that can be put on a line. Sometimes I think, "what a mess and what I am doing here?" But then, it is human destiny to lower entropy wherever we see a mess.

One, Two, Three, Many

Here is another example of using qualities in arithmetic.

One, two, three, many -- that is how, we imagine, prehistoric people were managing discrete quantities, before inventing what I called the stone age numerical system S_1. They probably knew rudimentary addition, observing 1+2=3. But one doesn't go too far in arithmetic without enough numbers. Of course they had high-brow shamans -- meta-mathematical theoreticians of that time -- and the shamans could know how to handle "many" within arithmetic but probably kept it as sacred secret. I wasn't born yet to explain to them that "many" is a quality and should not be manipulated as a quantity. So the rules probably went like this:

number plus many = many

many plus many = many

Cantor also was not born at that time to earn his title of shaman-in-charge-of-infinity. So he couldn't explain that there are different levels of "many," and couldn't present his table of addition and multiplication which for "many" would be almost like his table for alephs. (I know, it is bad manners to joke about dead people, but I can't help it. At least alephs are alive and still marching around in their purple socks. Well, not in Rome, of course.)

What Is A Line?

To avoid a possible case of mistaken identity: when I say "line" I mean a continuous entity of the first dimension used in Euclidean geometry, as opposed to a point which is of zero dimension. (Well, dimensions are the fancy latest stuff. Euclid simply stated that "a point is that which has no part" and "a line is a breadthless length." This says enough.) Who knows, all those smart people I disagree with may mean some other line with or without redefinition. Maybe the rule is: every one plays with his own line. As far as relations of line and numbers, there were greatly overexploited and, I would say, misexploited. There is even Cantor's axiom that declares the one-to-one correspondence of all numbers and all points on a line. Just like that, an axiom! Hold your unicorns, Mr. Cantor. Fantasy is good thing even in mathematics but in this case some attention should be paid to simple questions like what is a number; is there any relation between numbers and numerical systems; what is a line; and how did all that overwhelming infinity of points get there? Euclid certainly did not stuff a line with points. But then, I should go easier on Cantor. The knight of infinity was punished as much as a mathematician can be: by finding paradoxes in his own theory.

Lost my temper. Not sorry. It is simply that I am old-fashioned to the point of seeing axiom as something apparent, obvious, accepted by all reasonable taxpayers. I know, there is another school of thought: to declare as an axiom anything you want, weird or not, as part of the foundation of your theory, and then build you theory and see how it goes. Well, whatever followed from this one-to-one correspondence axiom did not go too well. It equated a denumerable set of numbers with an absolutely limitless infinite set of points that can be put on a line and then both, numbers and points, had to be squeezed into the continuum hypothesis. Not only did the continuum hypothesis not go well, but the huge machinery of a new logic had to be built to justify that it cannot be proved or disproved. I know there are probably things we may never know, but we should know about the pieces of knowledge we build ourselves, and should not hide behind a fence of unavoidable incompleteness.

Here I simply stated my understanding of a line based on common sense. I certainly cannot go through all definitions, almost definitions and descriptions of a line and numbers that mathematicians have made throughout the centuries. Who can number the clouds in wisdom? (Job, 38,37).

The Linear Continuum

The problem of uncountability of points on a line calls for a philosophical note. It is not for me to judge if the level of uncountability is really important in some areas of mathematics, but it is certainly an unfortunate custom to ascribe to this qualitative characteristic some quantitative meaning, and then try to evaluate the power of a set of points of a linear continuum. We have to remember that points on a line (when they are there) are discrete objects with a length equal to zero. It was wise of the creators of classical calculus to use some interval dx in order to fill an interval x with infinitely small intervals, so small that they can be viewed almost as points, yet they have lengths and for that reason they have a dimension of length. (It is an amazing feature of classical calculus that one leaves dx actually undefined as far as size goes, yet the theory works).

The continuity of a line as it is given in Euclid's geometry simply cannot be built from discrete points. We can, of course, define a non-Euclidean straight line that contains only so many points per inch, but it will be somewhat para-logical: we have to put those points along the line, so we have to have a line before we produce a non-Euclidean "pointy" line. That is not to say that I have anything against para-logicians socially. Some of my best friends are para-logical, and I even would defend their right to practice logic without a license.

In traditional geometry we can put a point on a line anywhere, and there is absolutely no limit to how many points can be put there. Cantor's scheme might take pride in the declaration that some infinities are more infinite than others, but for points which can be put on a line, any measure of infinity like $\aleph_k$ with any k, is a limitation. We will run out of alephs before we run out of holes on the line. The continuum hypothesis doesn't go so far -- it states that a linear continuum has only the first level of uncountability and the power of set of points in it is $\aleph_1$. And that is a limitation.

However, as far as a continuum hypothesis goes, that was a technical note about a non-existing problem. The line is the entity of the first dimension. Just because we can put points on a line does not mean that a line is made up of points. Indeed, we can put some number of pound pieces of cheese on a one-square-foot table, but it is nonsense to ask how many pounds there are in one square foot. Different dimensions dictate strong limits to our intellectual fantasy. One has to accept the limitations of freedom in order to be free. That is the paradox of social life, and also an example that living in peace with a paradox is not a tragedy at all. There are plenty of people who choose a more logically consistent way of life: either freedom without limitations, or limitations without freedom. Let them write their own logical satire. It could be fascinating.

Continuity And Points

Intuitively we imagine continuity as the state of a system when the elements are placed next to each other. If the elements of a line are not points but linear intervals, no matter how small, as we have now finally established after the reign of a dark century in meta-mathematics, then for continuity at least one point at the end of one element must be at the beginning of another. A line can be covered by intervals continuously. They have to have a common neighboring point. This is O.K. They don't fight over one point, which is zero size anyway.

If points and not intervals are elements of a continuous line, then for continuity the distance between neighboring points must equal 0, and because the linear size of a point itself equals 0 we have to conclude that overall the length of a line equals 0. That is fine with me if people prefer to play with such a small line -- not that they will know which direction it is pointing.

This is pure arithmetic. Countable or uncountable infinity cannot -- or at least should not -- shake definitions. For that reason, points cannot be elements of a line, and all the problems of counting the points of a line do not make sense without a redefinition of what zero is, what a point is, and what a line is. You want to redefine? I will not even ask why. Just don't affect good old arithmetic.

How Many Points On A Line?

Is a line made from points, or a might a line simply host the points? Euclid's axioms do not make this clear, as the geometric properties of a line do not depend on what it is made from. Euclid was certainly not a constructivist: he did not care to state what material he used to make his lines. For the needs of geometry it is enough to know, however, that we can put a point on a line or put a line through a point or two.

The question in the sub-title may sound silly, so let’s make a game out of it. Let’s ask a child armed with a pen and a ruler to draw one inch of line on a horizontal plane with vertical movements of the pen only. In other words, ask him to make a line from points. Let’s say the result will be 52 points. A thinner pen will bring results after 173 points. A smart child would ask us what will happen if there is an even finer pen tip, because the quantity of points to make an inch of line certainly depends on the size of the points. That child apparently is smarter than many mathematicians who believe that a line of finite length can be made out of points of zero size.

Despite the fact that one can imagine any quantity of points on a line, they are actually not there -- similar to the fact that there are no zeros within the number 7. We can add a quadri-zillion zeros to the number. We'll get the same number, and the next user of that number will not even notice our remodeling effort.

If we put points on a line, physically or in an abstract way, the line will host them in any quantity -- an absolutely unlimited set of points, be it $\aleph_1$ or $\aleph_{17}$. Agreeing on that would be a first step toward understanding that a line is not made of points, just like the body of a cube is not made from pieces of lines, and one is not made of zeros.

The belief that a one-dimensional object can be built out of an infinite number of zero-dimension objects is quite weird, but for reasons unknown to me the best minds for more than a hundred years did not notice it. In classical mechanics such weirdness is impossible. Indeed, any reasonable school child can figure out that if there are any quantity of particles, no matter how infinite the quantity, and the velocity of each particle is equal to zero, then altogether the particles will have zero kinetic energy. ("Classical" is the key word here.)

A line is a continuous one-dimensional entity of its own kind. Abstract points, lines, planes &c are primary objects, each of their own kind. They are not made of each other.

Maybe I should view the inability to understand that a line is not made from points, as a simple deficiency in abstract thinking -- nothing to be ashamed of. After all there are plenty of people who understand a number only if it is a number of something, not a number in general. Lucky them -- less troubles with general theories.

As to hosting points, there is no limit to how many points a line can host. If one put onto one inch of line a set of points of any power, and say "that's it, I covered that one inch of line," he will be wrong. There will be plenty of holes, one-dimensional holes, for more points. In fact, the combined length of those holes would be equal to the same inch we started with. This is easy to check. Take the set of zeroes of the same power as the said set of points, add those zeros all together and you will not get one as a result. The infinity of the set will not help.

If a young mind is ever bewildered as to what is left to do in the foundation of human knowledge, we should keep this example of covering a line with points as an illustration that human knowledge is built in disregard of many finer details in its foundation. Each successive generation can always find holes in what previously was proclaimed to be the continuous final truth. The main thing for young people on a road to finding such holes is not to trust their teachers too much.

That is an additional reason why arithmetic should be defended. I don't want teachers to lie and tell kids that .333...=1/3, justifying it by the concept of infinity -- which silly kids cannot even check. To accept such nonsense, kids have to trust their teacher, and that is the way to raise obedient conformists. What is more, that is the way to raise followers of those teachers.

It is actually all pretense when people fight over an election and political slogans. The actual rulers of a country are its teachers. They raise voters as they please. Jesuits use to say: “Give us the first six years of a child's life". Well, six years are not enough these days to stuff a child with all the prejudices. Brainwashing goes well into college years. Jesuits needed to fill poor kids’ heads only with fear of the Church, obedience to the Jesuits and the idea of an omnipresent Devil. We now live in a more complicated world, particularly having a whole aleph of devils.

Points And Intervals

Understanding that a line is a continuous, one-dimensional entity puts the problem of continuitum in a different light. It’s one thing to use tricks to assure continuity of a line which was erroneously presumed to be made out of points, and another to understand that a line cannot be anything but continuous.

What I call holes between inserted points are, of course, intervals of a line. They are entities of the first dimension.

They cover our one inch of line continuously. The end of one interval is the beginning of another, with the inserted point belonging to both -- unless some theoretical fantasy excludes that point from one or both intervals, in which case open intervals will be wrongly announced. There is some prejudice among mathematicians against bordering intervals, both being closed and having a common point. We should remember, however, that an interval does not get larger or smaller plus or minus one point. Open intervals on a numerical scale make sense, but on a line it simply does not exist. The highest authority on a subject pronounced: "The ends of a line are points." That means that talking about lines as Euclid pictured them as pieces of line -- not like we do now, thinking of a line going to infinity at both sides. Oops! I see now that I myself am susceptible to the foolish use of words. "Going to infinity" is actually a silly expression, and that is after I explained that infinity is just a negative characteristic that denotes the absence of an end. So, to clarify, there is no such place as infinity. I guess that if it did exist, it could be anywhere, including right here. In that case, I could have quite a marketable title for this work: "Letters From Infinity".

Now, back to intervals. Let’s do some counting. Say an inch of line can host c points, with c being what Cantor modestly used for the power of a linear continuum instead of $\aleph_1$, which was expressed as a hypothesis. (Actually, one may safely forget about modesty when dealing with infinity. I suspect his happy followers would accept it from the beginning if *c* would be declared to be $\aleph_1$. Who can count it anyway?) So, if Cantor puts c points on one inch of line, then with my reminder that the length of a point equals zero, there are *c* intervals between them (well, *c* less one interval, but again, who will count?). Now, surprise: all those intervals are finite! Indeed, if the point-putting procedure is done, no matter how many points you put there, the intervals are finite. Neat, huh? If one continues to pour points on the line through an infinity of time, then one might insist that intervals are on the way to approaching an infinitely-small state. So, this is actually a test on infinity: even if it is uncountable in the eyes of the devoted faithful, is it constant, is it done, or it is growing? There is not much clarity in the smart writing on this point at all.

If the infinite set of points on line is a constant, then all intervals between those points are finite, and the set of finite intervals on the line is certainly countable. From the basic premise that a line can be made only from intervals, it follows that no matter how many points we put on the line, they break the line into a countable set of intervals. That means that any set of points we put there -- a point being the end of interval -- is also countable. Here is your c, Mr. Cantor. It didn't manage to become $\aleph_1$. And we can save a lot of alephs. This just reinforces my explanation that countability is not a quantitative but a qualitative characteristic.

Mysterious sets can be declared uncountable until we figure out how to count them. Nothing to do with "how many". I bet you that, a presumably finite set of mosquitoes on Earth is uncountable, but if we would manage to put them all along a straight line (dead I hope!), the problem would be solved.

The countability of a linear continuum shows that if one need to look for uncountable infinite sets to preserve the self-respect and pride of the set-theorist, one should look elsewhere, not at points on a line. As to any other set, can it be uncountable or not? What do I know? I have not evolved enough to make peace with the idea of the uncountable. Maybe you should imagine an uncountable set of sheep, and check if you'll get to sleep quicker. Or, reminiscing from the days when I labored to understand the concept of entropy, maybe one should try to count indistinguishable particles in a statistical system. The reward could be great: once we labeled them 1,2,3... entropy might change.

Zeno's Legacy

Who started it, anyway, this prejudice that a line contains points even if we didn't put them there, or even that a line is made of points? Euclid didn't do it. The line in his axioms appears as an independent entity. There is some assumption that a point can't be on a line anywhere, but this is just a natural assumption, and doesn't mean at all that without an infinity of points there is no line. So, despite the ambiguity of the descriptive concept of “line,” there is nothing in Euclid’s axioms that would make people think that a line made from points.

It seems to be Zeno's legacy. He assumed that a time interval is made out of a series of instants -- no surprise that his arrow couldn't fly. His perception of line was also weird, as it would be made of points. Remember, Achilles did not take his normal steps in pursuit of the tortoise, but was carefully stepping only on certain points prescribed by Zeno. It is amazing how an ancient curse can still affect contemporary thinkers. But then, can we really be sure that we got much smarter? Should we expect more from our silly selves than from the smart Greeks?

Of course, the huge baggage of knowledge collected for more than two thousand years, since the Greeks started deductive science. But knowledge is like light: God created light and only then, after creation of it, separated light from darkness. Gradually, we do separate knowledge from nonsense, which we acquired from dead people together with knowledge, but that requires time and an inclination to doubt. A sad observation: humans in general are more inclined to believe than to doubt.

The rigorist technique grew enormously, too. Pages and pages of logical inferences would probably be as bewildering for Zeno as for me. But does one always know what to prove, as well as how to prove it? Judging by all the work done by very smart people in connection with the nonsensical continuum hypothesis, it looks like Zeno can still dance in his grave (or ash hill) -- he planted deeply the seeds of ignorance about continuity. I just hope he had as much fun fooling people as I have had trying to de-fool them -- without much hope, though.

Without detailed historical research, I have the impression that in the last two centuries this strange point-dependant perception of a line was supported by attempts to squeeze continuity out of a numerical scale. A line is continuous: that is assumed, or must be declared by definition. Yet no powerful set of points can make a line continuous without the line being there in the first place. A line has length; points don't. There will always be pointless holes -- intervals of a line free of intruding points -- holes between points on the line. You can make those holes smaller by putting more points on the line, but you'll simply increase the number of holes. In fact, it is impossible to put points next to each other without a hole between them. Rephrasing the famous saying of a female poet, a hole is a hole is a hole.

A line does not need points to be continuous. It just is. Without continuity of a line, life would be even more interesting. Imagine a triangle without angles: lines would go through the holes of each other without having common points. Euclid's axioms would be incomplete, because if one line goes through the hole of another line, they would not have a common point. You might as well call them parallel.

You cannot even interrupt a line by taking one point out. If you put that point there in the first place, you are taking back what is yours. If I put it there, I might let you take it out. If nobody put the point there, then the point is not there and you cannot take it away. Points are not the building blocks of a line. If you want to interrupt the line take out some interval, no matter how small. To put it plainly, a line is made from an abstract material other than points.

As to the numerical scale, hope for its continuity goes way back, but I think Dedekind was the most instrumental in developing what can follow from the belief in a one-to-one correspondence between real numbers and all points on a line -- and that is without analyzing what a numerical system can handle. In fact, he used a limit procedure to jump over holes. Strangely enough, continuity of both – a line and the numerical scale -- became an issue. As a result, mathematicians accepted that there are more numbers than can be expressed by numerical systems, and there are less points on a line than could be put there. An excessive estimation of the quantity of numbers came from using a qualitative characteristic -- countability -- as a quantitative. Underestimation of the ability of a line to hold any, absolutely any, number of points came from a misunderstanding about what a line is made of, and from the weird assumption that zero-size points can cover the length of an interval.

A word of caution. There is still an abundance of theoretically-defined "numbers" between those numbers expressed by a numerical system. We saw irrationals hiding in the cracks. We saw sin(1/x) near zero. We can construct plenty more. Don't rush to hope that they will provide a refuge for the dream of uncountability. Those "numbers" are the product of our abstract might. What we create, we know the location of. We know -- or should know -- how to count it. Uncountability is a presumed property of mysterious objects given to us. It is for "things in themselves," speaking in an old-fashioned manner. All our creative arrogance for millennia has been directed toward getting rid of such puzzles.

The Logic Of the End

I am making fun of some concepts, but it doesn't mean that they are necessarily wrong for all occasions. Well, some are, of course, but they are less wrong from a human point of view if they serve their purpose. Somehow, those who are supposed to enforce the law of the excluded middle make exceptions in cases when deviations are really, really needed. I once called this human approach to orderly thinking "the logic of the end": know your goal, then vary the rules of logic to support the needed inference. In both politics and litigation, it works like a charm.

Indeed, if we admit that we have a continuous line how do we humans assess the desired places on that line through numerical coordinates? With a discrete numerical system, sooner or later we will see some hole. So let's pretend that numerical system is overcrowded with numbers so dense that it can be declared continuous, whatever that property means. Having that pseudo-continuous system of numbers, we then assign them to assumed points on a line, and pretend that those points were there since God and Euclid created the line. Notice that the definition of a point as an entity that does not have a length, was never changed by any legitimate voting in the high shamanical council, so we assume that a line is covered by such an infinity of points that it is covered completely, even if the points do not have length.

Holes? There couldn't be any: there is an uncountable infinity of points!

And what is a continuous way to go from one number to another, if it is supposedly a continuous numerical scale? To go smoothly, without jumping over any holes? Well, those infinite decimal fractions are infinite enough to assure, somehow, a smooth transition from one number to another!

That was the actual fuzzy logic that covered the needs of differential calculus, and that relied on infinity as a medicine for many maladies.

Take note: both the numerical system and the continuous line were implicitly redefined in order to assure proper communication between then. As a result, the numerical scale is assumed to cover more numbers than it can, and the line is less continuous than if it were not made of points. (Well, an entity cannot really be less continuous unless one defines the degree of this property. A line made up of points is simply not continuous.)

Could the needs be covered without redefining both numbers and a line? Well, smart people were doing all that. I assume they knew what they were doing. I don't even have a clear picture why Cauchy's δ–ε formalism was not enough. Did ε and/or δ fall into the holes on a numerical scale? Even that wouldn't be a problem; one can courageously hide behind the concept of theoretically-defined numbers. (That is not my ε!)

What to do? I am here to criticize, to laugh, even to ridicule if things are getting too infinite, but I am not here to offer solutions. I am retired, after all. I simply will note that if one wanted to clear up this mess, then the acknowledgment or construction of the correspondence between *intervals* of a line and the *existing points* of a numerical scale might be unavoidable. Continuity can be achieved only if one element ends and another starts. Intervals can provide for that. Points can't. Actually, numbers cannot, either. Where is the end of one number and the beginning of another? Don't promise me that in infinity it can happen.

Misconceptions About Zero

Well, for a guy who doesn't know why he is doing it, I am doing O.K. so far. There are a few miscellaneous but finite points left to address, however.

The problems of infinity unavoidably bring me to that weird and multi-faceted creature: zero. Strictly speaking it is not a number. It is the absence of a number, and for that reason it can be viewed also as a quality.

Originally, numbers were used to count discrete objects. "Are there some objects?" If "yes," we can count them. If "no," we cannot count them. So the expression "there are zero objects" does not make sense, or it is a euphemism. (It is already stretching grammar if there is just one object and we use the plural in our question; but let’s be lenient about that.)

In geometry, a number is the result of a length-measurement procedure, and the protocol of that procedure is part of the definition. When we measure the length of an interval, it has to be the distance between *different* points. Otherwise, in a sneaky way we include entities of different dimensions in the definition of one object, risking complications. Length AB with A being not B is a number;

length AA is zero. AA is not an interval, does not have length and should not represent a number. One should try to be accurate about such things. This paragraph might give the impression splitting hairs, but at least a hair has some thickness.

Mathematical objects are interesting to deal with because they retain certain properties despite some transformations. Usually those transformation are reversible. Once we allow the length of an interval to become zero in the course of a transformation, any geometrical or numerical object loses its properties irreversibly. That is, at least, until the next leap of technology when someone will invent zeros with hidden meanings. Imagine a bouquet of flowers mathematically transformed into a point. One would look at such a point and treat it like nothing -- like zero, or even not notice it. But those who have a key to unlock that zero would secretly enjoy the color and aroma of the hidden flowers. Quite promising, actually. One could send flowers to one's "romantic partner," as they say these days, by e-mail in the form of a transcendental code-key to unlock the zero. Beam up the flowers, Scotty. Maybe it is funny, maybe not, but some topologists seriously stretch points into intervals. I saw it. I don't mind. They should try it with flowers or insects.

Instead of declaring zero to be a special constant, one can view it as the negation of a number, or a logical operator that turns a number into no number (by multiplication). If one would again get sufficiently loony to equate mathematics with logic, maybe zero should be the starting point of logistical mathematics, as it is certainly at the crossroad.

The beauty of logic is that it is object-independent. Mathematics, on the other hand, is dealing with certain classes of objects. Mathematics may pride itself on the fact that classes can be very wide, and the faces of individual objects do not play a role. Still, the rules of mathematics are far from the generality of logic. Of course, logic tries to study itself, but that is meta-logic, really.

In the Peano axioms, x+0=x means *number + no number = the same number.* Indeed, those axioms are for numbers only, and must contain a defense against intruders. Maybe a system needs an even more elaborate defense than now. I think it is an important safeguard for an axiomatic system. If one labors to build axioms of theory for manipulation with, let's say, natural fossils, and then some joker will throw a mummy into a pile of old bones, the system of axioms should be smart enough to defend itself. It was quite a lesson for all of us when Peano's system got indigestion from a *Concis* statement.

Even I can admit that I was carried away in these paragraphs with presenting various possibilities for zero. Let's go back to the traditional understanding of zero, considering all my objections overruled by His Royal Emptiness, the king of all zeros. However, even following tradition we will not see one traditional zero. They are quite different, indeed.

Small And Not So Small Zeros

In the theology of Alephs, the custom of characterizing sets by their countability or non-countability assumes some estimation of quantity. That reflects on the evaluation of the true value of zero. One cannot expect that people will refrain from asking the simple question: if the uncountable set of *c* points actually covers an interval, then what is the length of each point (or size of zero)? I could slightly stretch the usual arithmetical rules and treat the "power of continuum" *c* as a number and deal with 1/c or $1/\aleph_1$ and so on to evaluate corresponding zeros. But I will not do it: such division is not defined by people who are accustomed to play with those entities. But if some quantity of points can really fill a one-inch interval, then the question of the length of a point must be addressed. And, because the length of a point is zero, I will get an impression of the size of that particular zero. Apparently if c points cover a one-inch interval, then points have some size, so the corresponding zero 0_1 is skinny, but it is certainly not nothing.

But then, there is also 1/E. It cannot be the same zero as the previous one. Indeed, E is much much smaller than *c* if the power of an infinite set is an estimation of quantity. So 1/E is rather on the plump side, much more presentable then 0_1 indeed. Nevertheless 1/E (my ε) was declared to be zero in all textbooks I ever saw, and those textbooks recognize as a divine revelation the covering of an interval by *c* points. Fine, we have two zeros already, but then there is a third zero, which is the undisputed nothing. As far as I know, it never volunteered to cover any interval.

From the point of view of common sense, one might see no sense in that. Yet it is why we have smart theoreticians to dig deeper then blue-collar common sense would care to go. So if we really want to dig, the following questions are unavoidable: either you cover the interval [0,1] with an infinity of something, or not. And if you do, what is the length of that something? I cannot understand why Cantor -- who practiced galloping on the back of infinity so bravely -- couldn't see that his game simply invites playing with the other end of infinity, with the "cholera-bacilla" of infinitesimals, using his colorful, though tacky expression. You cannot play with an infinity of points covering line and declare, "let's play only with big things". The size of the elements from which you build your big toys sooner or later will demand attention. So let's sort our zeros.

1. 0_p -- the zero of the Peano axioms, the zero of common sense, which is a proud nothing perceived as a number,

which is 0=1-1. This is the same zero the lucky Greeks did not have.

2. 0_0=1/E -- the "countable" zero. We take a finite number and divide by countable infinity, be it the infinite n or 10^n. That is my ε and for that reason it shouldn't be a zero. That is my tool for counting the numerical "continuum" -- full-figured, but still pronounced to be nothing by those who proclaim .999...=1.

3. 0_1 -- the "uncountable" zero of the first kind. If uncountable infinity is much more infinite than countable, then this zero is much skinnier than the previous, yet is fleshy enough to provide points with the ability to cover a line. It has enough linear size to fill a finite interval if taken in good quantity.

Skinny and Plump Zeros

Those who count infinities by establishing a one-to-one correspondence, tell us that two circumferences of different diameter are made out of the same quantity of points. So the tiny circumference of a one inch diameter has the same quantity of points as the circumference having the same center and surrounding our Galaxy. (Disregard the fear that the circumference around our Galaxy is not completely round. That's an aria from yet another contemporary comic opera. Lets be Euclidian on this occasion).

Indeed, if we connect by an infinity of straight lines each point on the galactic circumference and its center, the lines will cross our small circumference each in a different point, and a one-to-one correspondence will be established (Fig 1). The size of the points from which the small circumference is supposedly made, I call skinny zeros. The points on the galactic circumference have to cover a much larger length so they must be ... well, fat, if I forget all political correctness. The more interesting question, however, is whether they are fat only along the line, or in all directions? I guess they must be roly-poly all around to please the people who prefer plump points, and who believe that planes made from lines, and space is made of planes.

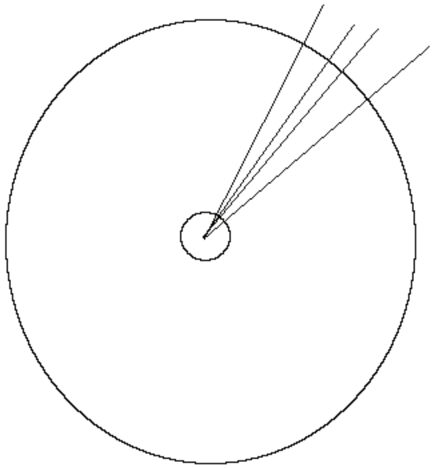

Fig.1

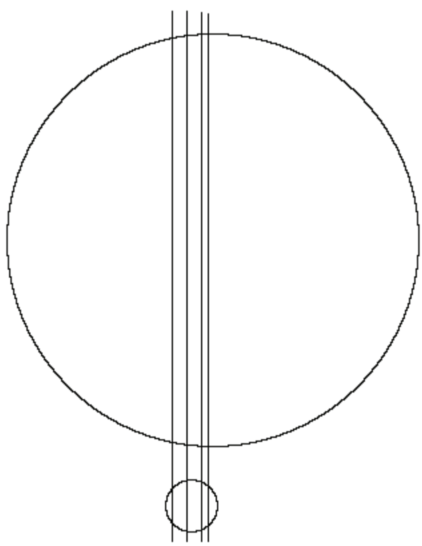

Fig.2

So why are the points on the galactic circumference so well-rounded? Is there are any physical law that does that?

Now let's create a set of parallel lines as shown in Fig.2 in such a way that each line crosses each circumference twice, and all points of a practically two-inch arcs on both sides of the large circumference are crossed, as well as all points of the small circumference. *Et voila*! We have a one-to-one correspondence again, although this time it is just between the small circumference and the four inches of the galactic circumference. How did it happen that points on four inches of galactic circumference have no reason to be more pot-bellied than on the small circumference? In fact, on the galactic four inches they must be even a little skinnier! What is more interesting is that the rest of the infinity of points of the large circumference remains either unacounted for, or nonexistent. It looks to me like a contradiction, unless that pesky "bend over" gravity has something to do with it.

This is a serious problem with counting uncountable infinite sets. "Uncountable" means only our inability to put 1,2,3... labels on the elements of the set. Yes, *our* inability or lack of knowledge how to do it, so it is a subjective characteristic. Counting points through establishing a one-to-one correspondence is supposedly not subjective. Once its done, one cannot say anything against it. A large circumference indeed has as many points as a small one. At the same time, four inches of the large circumference has the same quantity of points. I am sure I did not discover this contradiction, but many thinkers before me swallow this mishap without protest. Maybe they thought, "Well, it is infinity. Things happen there". I don't protest, either. I simply point out that each circumference is not made out of points at all. Rather, there as many points as we put there by our lines, by the power of our mind. And on the remaining part of the galactic circumference that is not crossed by our lines in second case, there are no points at all simply because we did not put them there.

Was it any help for you to sort out all those skinny and beefy points? Try and try again. The zero is in your court now.

Anti-Voodoo Axiom

For too long, infinity was viewed as some foggy area where anything might happen. It is rare when impartial people deal with infinity. The fact is that each mathematician assumes exactly those fairy-tale qualities of infinity that are favorable to his theory. We want to express transcendental numbers through common -- well, decimal -- fractions? Let's go to infinity. Somehow they will become the same there. They'll get united with their guarding rationals. We want to turn the first dimension into a zero dimension? Again, let's go to infinity (well, to the infinitely small, that is). Does someone want to get a sphere from an infinitely large circle? With the magical quality of today's mathematics, it can be done.

In old times there were at least poetic justifications for the human mind exploring infinity. It was the scene of divine copulation, of struggles between gods and titans, of reducing the entropy of real chaos (not chaos with hidden rules, as this days -- accept no substitutes!), or for creation of multiple worlds and whatnot. In fairness, I have to mention the special quality of the people of India, or more precisely their ancestors. If anything justifies the idea that some infinities are more infinite than others, it is the poetry of old Indian texts; but in the poetic world, this is healthy. (The talented Greeks gave birth to deductive science, but they were relatively primitive and too erotically pragmatic in constructing Gods. Pragmatism mixed with sex is really an old European malady.)

But the days of heroic poetic imagination are gone. We are down to Earth now even when we are exploring space. And in prosaic arithmetic, infinity should not be allowed to change the properties of things. Maybe it is time to agree on an anti-mysticism axiom about infinity, a statement like this: *the properties of objects do not change simply because the objects are viewed from an infinite distance.*

Having such an axiom would save us from assuming that a polygon might become a circle if its sides are very small. Effectively this is what people are assuming if they believe that π can be actually expressed and not just represented by infinite fraction. The intellectually-exciting one-man revolution of Archimedes -- calculating curves using straight intervals -- inevitably had to develop into differential calculus. But Archimedes wouldn't approve those who mistake a polygon for a circle simply because its sides are infinitely small. Such an anti-voodoo axiom would also protect us from trying to smash the borders between dimensions using the mystery of infinity.

Looking At the End Of Infinity

Logically, when we deal with potential infinity, endlessness in the mingling around numbers is much more comfortable than imagining a mysterious done infinity: you take a little step, do your business there and go to n+1. After a while you say "and so on." We know that these powerful words mean "forever", "to no end", "into a complete fog of endlessness". We cannot even dream to see the end of potential infinity; we're just happy that by establishing some properties of the perceivable part of the sequence, we can project the same properties into forever. Forever indeed, because potential infinity in our mind is connected with the mystery of time. What we see is the past of the sequence, really, even if we would count very far and reach the number U. By declaring that the sequence retains a certain property infinitely we, short-lived creatures that we are, leave our legacy to millions of future generations, which supposedly will keep witnessing that, for example, $n_{k+1}= n_k+1/2$. It is arrogance and vanity, of course, but still so pleasant. We are hugging God and asking him not to change the property of that sequence as it goes onto forever. But even that amount of vanity is not enough for some. There are those who want infinity here and now. So, the concept of actual infinity came to spiritually hungry mathematicians, shaking the logical foundations of their trade and giving them an unprecedented feeling of abstract power. I would say, "You've seen one infinity, you've seen them all." But no, in the drunkenness of their abstract might, they build elaborate structures using more and more infinite infinities as bricks. I don't mind. What's next?

In many ways actual infinity remains a logical puzzle. It also, however, has a great advantage in comparison with the cautious-sneaky-creepy-potential infinity: sometimes *we can see the end of actual infinity*. Indeed, the concept of potential infinity provide us with the knowledge of the arbitrarily large quantity of steps toward infinity, but the end of it cannot be seen and cannot even exist. Actual infinity is different. We assume its existence. It is "done" infinity. We will reach no end *if we'll walk stepping on the elements of an infinite set* -- just like Zeno's Achilles chasing tortoises -- but in many cases we can know what is at the both ends of actual infinity or in some parts of it. A paradox? Not really. Let's start with simplest example.

The actual infinite sequence of digits 1 starts with the digit 1 and ends with the digit 1. There is, of course, an infinite quantity of those digits in between, but the beginning and the end are known to us. So infinite is not always endless, only endless to count. This sounds funny, but that is the advantage of actual infinity. Contrary to that, with potential infinity we cannot say that the mentioned sequence of ones ends with the digit 1, despite the fact that there are no other digits. In developing infinity we defined the steps on the way to infinity, but we don't assume that infinity is done. We are still pleasantly tickled with the thought that it is somewhere further out there. The end of such infinity is always escaping from us, I would say by definition.

So, if actual infinity is assumed to exist, let's discuss any chosen part of it including the end. The set of points $\max\{\sin(1/x)\}$ in $[-\pi/2, \pi/2]$ is fascinating. It gives an example of a set of points with an infinity of them in the middle. We know both ends of it, and we can count the elements of this set from both ends toward $x=0$, so it is a denumerable set. This set helped us before to understand that my ε interval can host an infinity of finite intervals.

Now, here's an even more interesting example. Let's look at *all* infinite subsets of the infinite set of 0s and 1s. It is easer to picture it by trying to imagine all binary fractions between 0 and 1. As usual for fractions in a positional system, more significant digits are on the left (i.e. .1 is 1/2, .0001 is 1/16). Because we are discussing all subsets, there are some finite, some infinite. We concentrate on the infinite subsets within the concept of actual infinity, so let's present finite binary fractions as a finite part followed by an infinite quantity of 0s , for example .0001=.0001000... .

Now I say that if *all* subsets of the infinite set of 0s and 1s are under discussion, then the list of those subsets (presented as binary fractions from the largest to the smallest) must end with .000...001 preceded by .000...0010 preceded by .000...0011. That is simply because without those fractions the list would not be complete, and we wanted all the subsets. The infinity of zeroes is in the middle.

If one is still embarrassed to look at the rear end of infinity, let him accept my observations on the basis of symmetry considerations: if in the set of *all* subsets there is .100...000 (which is equal to the common fraction 1/2), then there must be .000...001. And of course .000...001 is my ε_2 -- the smallest number in a binary system.

Actually we should not be surprised that the Goddess of knowledge granted us the ability to know the end element of this infinite set of subsets. If we prefer to jump from potential infinity -- which is more or less digestible for the human intellect -- to the complete abstraction of actual infinity, we might as well have the tiny advantage of choosing how we build it, and in some cases to know the end of it. Infinity in the middle is still a mystery to us in this case.

There is something not healthy in combining the concepts of actual and potential infinity. I feel it but cannot put my finger on it. But if Cantor, Dedekind and their many followers want us to count (using potential infinity) infinite sets (which belong to the concept of actual infinity), then let's do it.

Counting binary fractions starting from a large fraction like .1 and then going to .01, 0.11 and so on is like counting the fruits on ever-branching trees. One would get lost in all those branches. So, if we know the end of an infinite set of binary fractions, let's start from the smallest fraction ε_2=.000...001. Then there is 2ε, 3ε and so on. Or we may even count from both ends like this: ε, 1−ε, 2ε, 1−2ε and so on. But I already did this in the system S_1. The set of fractions was countable then, and it is countable now.

On Dedekind's Proof of Uncountability

As I understand it, the belief in the uncountability of the points of which a line supposedly is made from, came about not actually from an attempt to count points, but from the wrong conclusion that the set of infinite decimal fractions is uncountable, and the wrong perception of a number as corresponding to every point on a line.

There is a known proof on the uncountability of decimal fractions in an interval [0,1] by Dedekind. (Cantor's proof is similar, but his first proof of uncountability of all points on an interval [0,1] dealt with an interval supposedly stuffed by points. In my way I already ridiculed that proof.)

So, Dedekind's proof goes: suppose someone found a sequence Q_n of decimal fractions that shows a way to count all those fractions. Let's construct a new fraction, D in which the first digit after the period will be different than the first digit of Q_1, the second digit will be different than the second digit of Q_2, and so on. In infinity we are supposed to get a fraction D that is not equal to any of those in Q_n. So, according to Dedekind, Q_n does not provide a way to count all decimal fractions.

Apparently this proof cannot work in the system S_1: there are no digits to alter as S_1 uses only the digit 1. This already puts Dedekind's proof in question, but I want to look at this problem from a different point of view: how we really can compare infinite sets without the use of qualitative characteristics.

There is an interesting curiosity in Dedekind's proof. Strangely, I did not see it discussed before. I would even come out with an *ad hoc* conspiracy theory if I weren't sure about Mr. Dedekind's high moral standards. The goal of his proof is to show that the set of fractions is uncountable while, obviously, the set of digits in each fraction is countable. I can start with the admission that there must be many more fractions altogether than digits in each fraction -- that is, if the words "more" and "less" were applicable in comparing infinities. The analogy with comparing finite numbers gives us the idea that in some yet-undefined way, the infinity of fractions is larger than the infinity of digits despite the fact that both are denumerable.

Dedekind's counter-proof is in decimals, but he wouldn't mind an anti-counter-proof in a binary system. He dealt with a matrix like the following, which presents my way of counting binary fractions starting from zero. Fractions Q_0, Q_1. Q_2 ... are written backwards to present their other ends. $Q_0 = 0$, $Q_1 = \varepsilon_2$, $Q_2 = 2\varepsilon_2$ and so on):

Q_0	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
0	1	0	1	0	1	0
0	**0**	1	1	0	0	1
0	0	**0**	0	1	1	1
0	0	0	**0**	0	0	0
0	0	0	0	**0**	0	0
0	0	0	0	0	**0**	0
0	0	0	0	0	0	**0**

Following Dedekind, we should build a fraction D which has the first digit from the end different than that in Q_0 , the second digit from the end different than that in Q_1, and so on (the digits to be chosen for change are in bold in my table). These are the diagonal digits as prescribed by Dedekind. It is clear that all the diagonal digits of my table are 0 "for a good part of infinity," so the end of fraction D = ...11111. The hope of those who put this uncountability proof into textbooks for second century is that if we are sufficiently immortal we will see a complete D. However, immortality will help us only to build the fraction D, but it does not mean that D will ever reach all Q_i. I don't say it will not; simply with this kind of matrix it is not obvious. It doesn't reach in finite examples, and it must yet be proven if in infinite cases it does. I'll show you why.

For the finite set of the power S, the power of the set of all subsets is 2^S. That means that the finite matrix similar to what was presented above is not square: there will be S rows and 2^S columns. So how will its diagonal will ever reach the opposite corner?

Can we see the same for an infinite set? Indeed we can. Consider the table

	1	1.1	2	2.1	3
1	1, 1				
2		2 ,1.1			
3			3, 2		

The left column consists of natural numbers n; the upper row consists of n each time followed by n+.1. For any finite n there are n rows and 2n columns. The diagonal elements A_{kk} are a pair of numbers: n=k from the left column followed by the k^{th} number from the upper row. As we see for k>1 the left number in a diagonal pair is always larger than the right number. Our diagonal will not hit the right lower corner of the finite table. That means that the numbering is not synchronized, so to speak, with the numbering sequence. In fact, we can compare the power of finite sets calculating the angle between the matrix diagonal as if it were a square on our table, and the actual diagonal of the rectangle.

Apparently this property does not disappear in infinity: the absence of the right lower corner does not change the angle of the diagonal of the infinite matrix. The matrix in this example is not a square by setting, finite or not, and "in infinity" the left number in the diagonal pair is always larger than the right number. The angular characteristics for comparing infinite sets might also be useful. I remind you that both sets, the one on the left column and the one of the upper row, are undoubtedly denumerable. You simply can go along the upper row and count the elements.

In Dedekind's counter-proof there is also not a square matrix, but target elements that form the fraction D are made from diagonal members of the matrix. In the finite form of such a matrix, if the quantity of rows reaches 100, then the quantity of columns reaches 2^{100}. So the finite analogue of Dedekind's fraction D cannot reach more then 100 fractions because it is dealing with diagonal elements.

Yes, we should go "to infinity" with this matrix. But what will change? Infinity is not a fairy tale where things change their properties and lead to miracles. If there is a matrix Q_{ik} - the diagonal members will always be Q_{ii} and never even $Q_{i,i+1}$ despite the fact that i goes to infinity.

Here is a situation when one must make up his mind: either infinity is constant, or it is running. If it is constant, it seems to me that the matrix for Dedekind's fraction is clearly a non-square -- if anything about infinity can be clear. If it is a running infinity, then how can we picture it? For each added row the matrix gets twice wider, that is clear. And for each added row the fraction D gains one element only and does not advance to the right too much. In both cases I don't see how D can cover all Q_i.

So, the fact that Dedekind's fraction is constructible has to be proven, not assumed. Independent of such proof, one might go along the upper row and count the fractions as we add my epsilons one after another an infinite number of times.

Apparently Dedekind relied too much on the assumption of Cantorian theory that the sizes of infinite sets can be compared only by the level of countability, despite the fact that infinite sets of intuitively different "size" have bewildered people since Galileo.

Countability is our ability to count; or, at best, a set's ability to be counted. It cannot replace a quantitative assessment of the set. Looking into the infinite non-square matrixes can help to uncover interesting things about infinite sequences. It provides some ways to compare the "size" of infinite sets. It is not actually size, you know; they are still of "the same size" in a way, but only because size characteristics are not applicable. It is more like comparing the activity of infinities, if you must do it. Maybe we should just say one is crazier than another?

Where Is Infinity?

Infinity -- and the extent to which we can understand it and operate with it -- is actually a question for philosophy. I believe in the power of pure reason very much. I doubt only humans' ability to diagnose pure reason when they accidentally use it. However, there is no way we can deal with subjects on the border between the facts of reality and abstract reasoning, without choosing a philosophical position. Numbers originally are not a product of pure reason but the result of our observation of reality. To be more exact, the existence of *discrete objects* is reality, and the concept of numbers is a result of the cooperation between reason and experience. The notion of continuity is also the result of experience, but that notion is much poorer in us. But just imagine that we know only continuity: imagine that we are liquid, as is everything around. If we still would be able to think, the only way to observe numbers would be to count our thoughts unless they would "liquidly" flow one into another. In that environment I would accept the uncountability even of finite sets.

The unavoidable philosophical declaration in arithmetic is actually about the limits of infinity. I cannot dictate my position to those who think that some infinities are more infinite then others. What I try to do here is to demonstrate that one infinity if quite enough if one must have it. I do it out of the belief that the infinite includes not only all that is finite, but also all that is infinite. It looks like I am disagreeable even when I try to be nice.

Once we learned that a line is not made out of points, that it is for us to put a point there or not (physically or mentally), then an interesting question leaps out at us: where is the infinite which would be *given* to us and not simply be a product of our minds saturated with ideas we are already bored with? It is still our potential and our choice to put as many points as we choose, but they are not there to give an example of actual infinity independent of our action. Forget the physical world -- we simply don't know if there is any infinity.

Still, in our abstract exercises, where is the object that actually contains an uncountable infinity of something? The set of all sets? Even if it is an object, thank God it is not an arithmetical object. (Of course, a set that includes itself in addition of its elements is automatically infinite, so people with a fixation on infinity still have toys.)

What is really infinite is our potential to put points on a line, add numbers to a sequence, to build any large sets. All that can be handled with the concept of potential infinity. But after the explanation that a line is not made of an infinite set of points, I don't see anything that would require us to accept actual infinity simply because it is there and it is not under our control.

I know it is difficult to give up one's favorite intellectual toys, but why not try? Let's say, try to live one year without actual infinity and without putting it into the heads of students. Maybe the U.N. should announce The Year Of Potential Infinity -- voluntary, of course; they don't have enforcement power anyway. We might get a little more finite, or get bored by the end of that year, but then the herd of humans might also discover that the grass is as green on this side of the fence.

That was about quantity connected with infinity in one way or another. As to uncountability, if it is not used as a quantitative characteristic, it is all a different matter. One doesn't get too recursive if one doesn't know how to proceed with counting. But then, it is not fun to joke about recursion; it is too methodical.

Renunciation

It is amazing how tolerant I am to the religious toys of others. "Know yourself!" This is my chance to learn the real scope of my acceptance of other people's right to preach whatever they choose. So many times during this writing I wanted to scream: "all those infinities, one riding on the back of another, it is all complete nonsense!" But then thoughts like "Well, maybe some like it, or need it, or maybe one even sees it through special foggy glasses or foggy ganglia", such thoughts stopped me. Does it mean that I am becoming a conformist? People change, you know! When I was young, my typewriter did not have the exclamation-mark key, and I did not need it for writing about things that normal people perceive much more emotionally than arithmetic. Now look how emotional I have become!

The same with conformism. What is stopping me from renouncing infinity altogether before God and Satan? I should simply say: there is no end to a natural sequence, but there is no infinity. Well, I really don't want to hurt anyone, especially those who believe that they can create new worlds simply by definitions. Let me be partially uncompromising and renounce the infinitely small.

Indeed, the annoying quality of infinity as it was understood probably since Aristotle is its fuzziness. Remember the ω-trick? We cannot trust infinity to stay the same if there is annihilation. Infinity is perceived either as:

1. We can add anything finite or even compatibly infinite to it; or

2. Perceived it as variable.

The first problem is not too bothersome if infinity is not mixed with numbers in arithmetic. What do I care if in geometry an infinitely remote point will become even more remote? Interesting results will not suffer. We might decide to be accurate and not recognize the $1+\omega=\omega$ annihilation even philosophically. The diagonal way to compare infinities which I showed in connection with Dedekind's proof can be very sensitive: it can notice even one column added on the left. Such accuracy might give us infinity we can trust a little more (still, be careful). Once we agree not to allow infinity to be too fuzzy, we will be less inclined to expect that anything could happen in infinity. It would improve the philosophical reputation of infinity, even with its virtue already tarnished forever. However, treating infinity as a constant might be difficult still. Somehow in every attempt to declare infinity to be actual, the idea of the potential, developing infinity is present in the back of people's minds. It is not healthy! For that reason the use of potential infinity as such is more frank, so to speak.

"Infinitely small" in the classical analysis is a kind of variable that gets smaller and smaller and never disappears. Now, I should feel guilty because once I got irritated with a student I was privately tutoring. I asked: “Give me an example of a value that gets smaller and smaller but never reaches zero." He gave it considerable thought and said: "A pencil. It gets shorter and shorter as we sharpen it, but it never disappears." I was young and silly, not recognizing that this is how normal people should perceive overly-clever mathematical constructions. *Vox populi* some time shakes great empires, not just petty abstract concepts.

Getting smaller and smaller is fine, of course, when you deal with variables in the first place. But what if we want to stop and look at it as if it were a number, even if only a theoretical one? How small and how infinitely small is it?

The perception of infinitely small as a diminishing variable was quite deep in people's minds through the centuries. Yet, an example that finite intervals can be hosted by the traditionally infinitely small interval 1/N with N going to infinity might give us the idea that there is no such thing as an infinitely small interval at all. With N representing a developing infinity, one would have trouble to define a *constant* small interval corresponding to N. But my example with the set M provides the possibility to say: "So what" -- the interval is infinitely shrinking, but every time on the way to getting smaller it still has the ability to hold finite intervals. That would certainly help us not to mix it with classical infinitesimals, those ghosts of non-conceived numbers, which are supposed to be smaller than any finite interval by definition. But those who made this definition, no matter what geniuses they were indeed, did not know how small a finite interval can really become; and they certainly did not have a clear picture of an infinitely small interval. One has to be a Catholic priest to offer advice on marriage without knowing what it is. But on infinity everyone can have opinion -- even I do -- and no one has seen it.

What I am doing here is declaring that a finite interval can be small to any degree. I am deciding not to assume that if it is finite, then it is not too small. So, I did it, I renounced the infinitely small. Amen. The problem with writing satire is that one becomes unsure if one is joking.

Creation Of Humans

I am back to Kronecker's aphorism. The attempt to understand the intellectual produce of humans is also the study of human behavior. Why was this or that function was taken as an example in some theoretical presentation, why is there a unanimous non-acceptance of some view, or what is the reason for fascination with this or that object of study? If one pays attention to such things, one notices that the intellectual life of humans is just the life of humans. They are hierarchical in their behavior, they are mostly group dependent, not to say corrupt in one way or another, and they may start bitter fights around the interpretation of some abstract object as if it were a piece of land with vital resources. Mostly, intellectuals belong to a human subspecies of those who seek immortality in the form of published works, but they are too modest to admit it. Of course, I plan to publish this unworthy writing too, but I am so busy with my interesting life that I can disregard thoughts about the infinity of uncountable postmortem glory.

All that is fine. Occasional examples of "out of this world" individuals who really care almost exclusively about knowledge provide enough inspiration for idealists. But then, idealists have a tendency to follow some complex of ideas and can get ugly toward those who prefer other ideas. That leads to ideological corruption in defense of one's camp. Some intellectuals can be corrupted in a common way, taking bribes in some form for supporting or disapproving certain concepts. Generally, it is known that corruption blossoms in any part of human activity that is not the subject of public scrutiny, and mathematics seldom is. Imagine politicians subject only to the control through peer reviews. I saw it in my young days.

Ideological corruption does not require a monetary transaction or an exchange of favors. One may submit himself to some ideological self-control for, as humans say, "higher reasons" -- not that it is always clear what those reasons are. Maybe they fear to rock the boat. Indeed, building knowledge is rarely done on firm ground: too much shaky suppositions are in the foundations of most areas of knowledge. Uncontrolled thinking can be dangerous, even scary for the thinker himself. As Anatole France's character said "Where are you leading me, my thought!"

All this lamentation is my attempt to understand why it is so important for very smart people to treat my favored decimal fraction $\varepsilon=1-.999...$ like nothing, to put it down, to deny its existence. They needed to get rid of it so badly that they even had to manipulate arithmetical calculations with the ω-trick.

Well, ideological corruption is here, I think. That tiny little number can really shake the boat. That is because fractions like .999... are not irrationals yet they are infinite. Loyalty to an initial concept becomes more important than the anticipated intolerance to nonsense that comes from attempts to defend that concept. Those who sanctified the unity of the trinity didn't care to notice that they were accusing their God of committing incest. Those who pressed the idea of equality on other people didn't care if they had to be unequal dictators to promote that equality. And in our politics, the deed is often less damaging than the cover-up that follows.

There is more to it. Periodic fractions stand in solitude. Irrational numbers are presented by infinite decimal fractions and don't walk into the depth of the numerical scale by themselves, only between two guards -- infinite decimal fractions of an unclear nature. As the brave English hunters in a perverted attempt to preserve tradition chase a poor fox until it has no choice but to end up in the box, those guarding rational decimals narrow irrational numbers down to their supposed place. Following instructions provided by the one-point theorem, two rationals, r_-, r_+ with *a* in between miraculously follow into the depth of the infinitely close and become *one* irrational number. What happens with r_-, r_+? How do they disappear in this new unity of the trinity? Actually, nobody cares. What are two miserable good-for-nothing numbers, even if they were born rationals (probably bastards, anyway)? The continuity of the numerical scale is at stake.

And then come periodic fractions. They arrogantly walk by themselves into the depth of the numerical scale without guards. The infinity of depth is reserved for irrationals -- how dare they? Besides, we know from their dossiers that they are supposed to be finite if they are rationals, at least in a different numerical system. So, to save the more important concept we should declare that periodic fractions are actually finite (well, infinite yet finite -- up to a point, that is) and annoying little things like ε_{10} we should declare to be nothing, zeros, hiccups of illusion.

My very good friend, knowledgeable in mathematical logic, said to me: "If that number exists, of course it equals zero". So much for an alternative. If I would be that ε, what is more depressing: to be zero or not to exist? May be proud non-existence is a nobler state. Nevertheless I am trying to pull my ε out of meta-mathematical Nirvana into the prose and discomfort of ordinary existence. Actually, that friend and another fellow, a friend of a friend, helped me in this work in its preliminary stages by dismissing the whole subject altogether to my gratitude: the strong disapproval of professionals is inspiring. It works every time.

In addition to general human curiosity about ways to formalize knowledge, I have a personal interest in showing that numbers and their sequences are countable. When my Vermont life-time fishing license expires and I join the ghosts of other disobedient thinkers in a warmer climate, they could shame me by saying: "Up there, you knew that Godel badly insulted arithmetic by spreading the seeds of social distrust in this noble science, and you did nothing about it." At my age one should think about such things.

Benson, Vermont, January 2005